Online Appendix

Appendix A. Proofs of Section 3

Appendix A.1. Proof of Proposition 1

Step 1. We begin by first proving the necessity of these conditions. The resource constraint (7), the maximum tax constraint (8), and the household participation constraint (9) must be satisfied by feasibility and by the fact that, in choosing their level of investment, households can always choose $i\left(z_t\right) = 0$ forever which provides them with a utility of at least $u\left(\omega\right) / \left(1-\beta\right)$. The necessity of the truth-telling constraint ($AS-IC$) follows from the fact that conditional on (q_t, z_t, θ_t), the policymaker can choose the taxes appropriate for $(q_t, z_t, \widehat{\theta})$ for $\widehat{\theta} \neq \theta_t$ and he can follow the equilibrium strategy from $t+1$ onward. From the policymaker's budget constraint (4), this provides him with immediate rents equal to $x_t(q_t, z_t, \widehat{\theta}) + \theta_t - \widehat{\theta}$ and his continuation value from $t+1$ onward equals $V_{t+1}(q_t, z_t, \widehat{\theta})$. Condition ($AS-IC$) guarantees that this privately observed deviation is weakly dominated. The necessity of the limited commitment constraint ($C-IC$) follows from the fact that conditional on (q_t, z_t, θ_t), the policymaker can choose to tax the maximum which from (4) provides him with rents equal to $f\left(i_t\left(q_t, z_t\right)\right) + \theta_t$. Given that $v\left(0\right) \geq \underline{V}\left(1-\beta\right)$, his continuation value from $t+1$ onward following the deviation must weakly exceed $\underline{V}$. Condition ($C-IC$) guarantees that this deviation is weakly dominated.

Step 2. For sufficiency, consider an allocation which satisfies (7) – (9). Since feasibility is satisfied, we only need to check that there exist policies so as to induce households to choose the level of investment $i_t\left(q_t, z_t\right)$ at every (q_t, z_t). Suppose that conditional on θ_t, the government sets taxes equal to one hundred percent if a household has not chosen the prescribed investment sequence up to and including date t. Otherwise, if a household has chosen the prescribed investment level, the government sets taxes equal to $x_t\left(q_t, z_t, \theta_t\right) - \theta_t$ for each θ_t, where this is feasible given (8). Given this tax structure, any investment level for households other than $i_t\left(q_t, z_t\right)$ is strictly dominated by investing 0 forever which yields $u\left(\omega\right) / \left(1-\beta\right)$. From (9), investing $i_t\left(q_t, z_t\right)$ weakly dominates investing 0, so that the allocation satisfies household optimality.

We now verify that the allocation is sustained by equilibrium strategies by the incumbent policymaker and the representative citizen. Suppose that following a public deviation by the policymaker at t, the representative citizen replaces the incumbent at $t+1$ for all realizations of z_{t+1}. Moreover, following any public deviation by the representative citizen, the equilibrium allocations from $t+1$ onward are unchanged. We now verify that the allocation is an equilibrium. We only consider single period deviations since $\beta < 1$ and since continuation values are bounded. Let us consider the incentives of the policymaker to deviate. Conditional on (q_t, z_t, θ_t), the policymaker can deviate privately or publicly. Any private deviation requires the policymaker to choose policies prescribed for $(q_t, z_t, \widehat{\theta})$ for $\widehat{\theta} \neq \theta$. This provides him with immediate rents equal to $x_t(q_t, z_t, \widehat{\theta}) + \theta_t - \widehat{\theta}$ and his continuation value from $t+1$ onward equals $V_{t+1}(q_t, z_t, \widehat{\theta})$. Condition ($AS-IC$) implies that this privately observed deviation is weakly dominated. Alternatively the policymaker can deviate publicly. Since all public deviations yield a continuation value $\underline{V}$

at $t+1$, the best public deviation maximizes immediate rents, and this is achieved with a one hundred percent tax. This yields rents equal to $f(i_t(q_t, z_t)) + \theta_t$ at t and a continuation value $\underline{V}$ from $t+1$ onward. Condition ($C-IC$) guarantees that this deviation is weakly dominated. Now let us consider the incentives of the representative citizen to not deviate. If he deviates from the replacement decision, the continuation equilibrium is identical as if he had not deviated. As such, his welfare is independent of the replacement decision, and for this reason any deviation is weakly dominated.■

Appendix B. Technical Results

In this section, we prove technical results regarding $J(V)$ which simplify our analysis. It is convenient to restate the result of Lemma 1 in Lemma 5 and Lemma 6, which we state and prove next.

Lemma 5. *$J(V)$ satisfies the following properties: (i) It is weakly concave in V, (ii) it satisfies $J(V) = \overline{J}$ for $V \in [\underline{V}, V_0]$ and it is strictly decreasing in V if $V \in (V_0, \overline{V}]$.*

Proof. Proof of part (i). Consider two continuation values $\{V', V''\}$ associated with corresponding solutions α' and α'' which provide welfare $J(V')$ and $J(V'')$. Define $V^\kappa = \kappa V' + (1-\kappa) V''$ for some $\kappa \in (0,1)$. It must be that

$$J(V^\kappa) \geq \kappa J\left(V'\right) + (1-\kappa) J\left(V''\right) \ \forall \kappa \in (0,1) \tag{Appendix B.1}$$

establishing the weak concavity of $J(V)$. Suppose this were not the case. Define α^κ as follows:

$$\alpha^\kappa|_z = \begin{cases} \alpha'|_{\frac{z}{\kappa}} & \text{if } z \in [0,\kappa) \\ \alpha''|_{\frac{z-\kappa}{1-\kappa}} & \text{if } z \in [\kappa, 1] \end{cases},$$

where $\alpha^\kappa|_z$ corresponds to the component of α^κ conditional on the realization of z, and $\alpha'|_z$ and $\alpha''|_z$ are defined analogously. Since α' and α'' satisfy the constraints (13) − (18), α^κ satisfies (13) − (18) and it provides continuation value V^κ, achieving a welfare equal to the right hand side of ($Appendix B.1$). Therefore, ($Appendix B.1$) must be satisfied since $J(V^\kappa)$ must weakly exceed the welfare achieved under this feasible solution.

Proof of part (ii) We first prove that $J(V)$ is weakly decreasing in V. Suppose by contradiction that $J(V') < J(V'')$ for some $V'' > V'$ where V' and V'' are associated with corresponding solutions α' and α'', respectively. Define $\widehat{\alpha}'$ as follows:[36]

$$\widehat{\alpha}'|_z = \begin{cases} P(z) = 0 & \text{if } z \in [0, (V''-V')/(V''-\underline{V})) \\ \alpha''|_{\frac{z-(V''-V')/(V''-\underline{V})}{1-(V''-V')/(V''-\underline{V})}} & \text{if } z \in [(V''-V')/(V''-\underline{V}), 1] \end{cases},$$

[36]Note that if $P(z) = 0$, then the values of $i(z)$, $c(\theta, z)$, $x(\theta, z)$, and $V'(\theta, z)$ are payoff irrelevant since households receive $\overline{J}$ and the replacement policymaker receives V_0.

$\widehat{\alpha}'$ satisfies (13) – (18) and provides continuation value V' so that it satisfies the promise-keeping constraint (12), and it achieves household welfare equal to

$$\frac{V''-V'}{V''-\underline{V}}\overline{J}+\frac{V'-\underline{V}}{V''-\underline{V}}J\left(V''\right)\geq J\left(V''\right)>J\left(V'\right)$$

where we have used the fact that $\overline{J} \geq J(V)\ \forall V$ by definition. This contradicts the fact that α' is a solution to program P_0. Now note that $J(V) = \overline{J}$ for all $V \in [\underline{V}, V_0]$ since by definition, $J(\underline{V}) = J(V_0) = \overline{J} \geq J(V)$ and since $J(\cdot)$ is weakly concave.

Note that by definition of V_0, $J(V) < J(V_0) = \overline{J}$ if $V > V_0$ since V_0 must represent the highest continuation value that the policymaker can receive conditional on the households receiving their highest continuation welfare $\overline{J}$. Given that $J(V)$ is weakly concave, this means that $J(V)$ is strictly decreasing in V if $V \in \left(V_0, \overline{V}\right]$. ■

We now move to prove the continuous differentiability of $J(V)$ for $V \in \left(\underline{V}, \overline{V}\right)$. From part (ii), we only need to consider continuation values $V \geq V_0$, since otherwise $J(V) = \overline{J}$ and $J'(V) = 0$. The below preliminary result implies that there is no turnover if $V \geq V_0$.

Lemma 6. *The solution to program P_0 satisfies the following properties: (i) If $V \geq V_0$ then $P(z) = 1\ \forall z$. (ii) For $V \in (\underline{V}, V_0)$ there is a solution with $0 < \int_0^1 P(z)dz < 1$.*

Proof. Proof of part (i) We first establish that $V_0 > \underline{V}$. This is because from the limited commitment constraint (16),

$$V_0 \geq \sum_{n=1}^{N} \pi(\theta^n)\, v\left(f\left(i(z)\right)+\theta^n\right)+\beta\underline{V} > v(0)+\beta\underline{V} \geq \underline{V}.$$

Consider the solution α given $V \geq V_0$ and suppose that by contradiction $P(z) = 0$ for some positive measure z. Define $q = \int_0^1 P(z)\,dz \in (0,1)$ and

$$V_q = \frac{\int_0^1 P(z)\left[\left(\sum_{n=1}^{N}\pi(\theta^n)\left(v\left(x(\theta^n,z)\right)+\beta V^{+}(\theta^n,z)\right)\right)\right]dz}{q}.$$

V_q corresponds to the continuation value to the policymaker conditional on preserving power. It is clear that since $V \geq V_0 > \underline{V}$, the promise-keeping constraint (12) and $q < 1$ imply that $V_q > V \geq V_0$. The weak concavity of $J(\cdot)$ implies that

$$J(V) \geq (1-q)\,\overline{J} + qJ(V_q)\,. \qquad \text{(Appendix B.2)}$$

Moreover, if it were the case that ($Appendix B.2$) were a strict inequality, then this would imply that the solution α conditional on $P(z) = 1$ provides a continuation value to the policymaker V_q and yields welfare to households strictly greater than $J(V_q)$, which contradicts the fact that the solution to program P_0 subject to $V = V_q$ is optimal. It follows that ($Appendix B.2$) holds with equality so that

$$J(V) = (1-q)\,\overline{J} + qJ(V_q) < \overline{J}. \qquad \text{(Appendix B.3)}$$

Define $\widetilde{q}$ as the value which satisfies $V = (1-\widetilde{q})\,V_0 + \widetilde{q}V_q$. It is clear that $\widetilde{q} < q$ since $V_0 > \underline{V}$. The weak concavity of $J\,(V)$ implies that

$$J\,(V) \geq (1-\widetilde{q})\,J\,(V_0) + \widetilde{q}J\,(V_q) = (1-\widetilde{q})\,\overline{J} + \widetilde{q}J\,(V_q)\,,$$

which contradicts ($Appendix B.3$) since $\widetilde{q} < q$. This establishes that if $V \geq V_0$, then $P\,(z) = 1$ $\forall z$.

Proof of part (ii) Consider any $V \in (\underline{V}, V_0)$. Choose α' with

$$\begin{array}{ll} P\,(z) = 0 & \text{if } z \in [0, (V_0 - V)\,/\,(V_0 - \underline{V})) \\ P\,(z) = 1 & \text{if } z \in [(V_0 - V)\,/\,(V_0 - \underline{V})\,, 1] \end{array},$$

and for all z, let the continuation equilibrium allocation be associated with the solution at $V = V_0$. Note that α' satisfies (13) – (18) and provides continuation value V so that it satisfies the promise-keeping constraint (12), and it achieves household welfare equal to $\overline{J}$. This implies that the strategy α' is a solution with $0 < \int_0^1 P(z)dz < 1$. ■

We establish the following preliminary lemmas. Define $C_{n,n+k}$ as follows:

$$C_{n,n+k} = v\,(x\,(\theta^n, z)) + \beta V^+\,(\theta^n, z) - v(x(\theta^{n+k}, z) + \theta^n - \theta^{n+k}) - \beta V^+(\theta^{n+k}, z).$$

The following lemma characterizes the set of allocations α defined in (11) which satisfy the constraints of the recursive problem (12)–(18). This lemma simplifies the problem by illustrating which set of constraints are redundant and can be ignored in different circumstances. To simplify notation, we let $\alpha|_z$ correspond to the component of α conditional on the realization of z.

Lemma 7. *For a given allocation $\alpha|_z$, the following must be true:*

1. *If $\alpha|_z$ satisfies (15), then $x\,(\theta, z) - \theta$ is weakly decreasing in θ and $V^+\,(\theta, z)$ is weakly increasing in θ,*
2. *If $\alpha|_z$ satisfies $C_{n+1,n} \geq 0$ and $C_{n,n+1} \geq 0$ $\forall n < N$, or if $\alpha|_z$ satisfies $C_{n+1,n} = 0$ and $x\,(\theta^n, z) - \theta^n \geq x\left(\theta^{n+1}, z\right) - \theta^{n+1}$ $\forall n < N$, then $\alpha|_z$ satisfies (15) $\forall\theta$,*
3. *If $\alpha|_z$ satisfies (15) $\forall\theta$ and (14) for $\theta = \theta^1$, then $\alpha|_z$ satisfies (14) $\forall\theta$,*
4. *If $\alpha|_z$ satisfies (15) $\forall\theta$, (14) $\forall\theta$, and (16) for $\theta = \theta^1$, then $\alpha|_z$ satisfies (16) $\forall\theta$,*
5. *If $\alpha|_z$ satisfies $V^+\left(\theta^1, z\right) \geq \underline{V}$ and (14) holding with equality for $\theta = \theta^1$, then $\alpha|_z$ satisfies (16) for θ^1, and if $\alpha|_z$ satisfies $V^+\left(\theta^1, z\right) \geq \underline{V}$ and (16) holding with equality for $\theta = \theta^1$, then $\alpha|_z$ satisfies (14) for θ^1.*

Proof. Proof of part (i). Note that the constraints that $C_{n,n+k} \geq 0$ and $C_{n+k,n} \geq 0$ for $k \geq 1$ together imply:

$$v(x(\theta^{n+k}, z)) - v(x(\theta^{n+k}, z) - (\theta^{n+k} - \theta^n)) \geq v(x\,(\theta^n, z) + \theta^{n+k} - \theta^n) - v\,(x\,(\theta^n, z))\,,$$

which given the concavity of $v(\cdot)$ can only be true if $x\left(\theta^{n+k}, z\right) - \theta^{n+k} \leq x(\theta^n, z) - \theta^n$. This establishes that $x(\theta, z) - \theta$ is weakly decreasing in θ. Given this fact, it follows that for $C_{n+k,n} \geq 0$ to hold, it is necessary that $V^+\left(\theta^{n+k}, z\right) \geq V^+(\theta^n, z)$.

Proof of part (ii). This is proved by induction. Suppose that $C_{n+1,n} \geq 0$ and $C_{n,n+1} \geq 0$ $\forall n < N$. From part (i), this implies that $x(\theta^n, z) - \theta^n \geq x\left(\theta^{n+1}, z\right) - \theta^{n+1}$, which given the concavity of $v(\cdot)$ implies that

$$v(x(\theta^{n+1}, z) + \theta^{n+2} - \theta^{n+1}) - v(x(\theta^{n+1}, z)) \geq$$
$$v\left(x(\theta^n, z) + \theta^{n+2} - \theta^n\right) - v(x(\theta^n, z) + \theta^{n+1} - \theta^n).$$

Together with the fact that $C_{n+1,n} \geq 0$ and $C_{n+2,n+1} \geq 0$, the above condition implies that $C_{n+2,n} \geq 0$. Forward iteration on this argument implies that $C_{n+k,n} \geq 0$ for all n and k for which $n + k \leq N$. Analogous arguments can be used to show that if $C_{n,n+1} \geq 0$ for all $n < N$, then $C_{n,n+k} \geq 0$ for all n and k for which $n + k \leq N$.

Now suppose that $C_{n+1,n} = 0$ and $x(\theta^{n+1}, z) - \theta^{n+1} \leq x(\theta^n, z) - \theta^n$ $\forall n < N$. Then this implies that $C_{n,n+1} \geq 0$ for all $n < N$, and given that this is the case, the same arguments as above can be applied. To see why, suppose instead that $C_{n,n+1} < 0$. Together with the fact that $C_{n+1,n} = 0$, this would imply that

$$v(x(\theta^n, z)) - v(x(\theta^{n+1}, z) - (\theta^{n+1} - \theta^n)) < v(x(\theta^n, z) + (\theta^{n+1} - \theta^n)) - v(x(\theta^{n+1}, z)),$$

from concavity of $v(\cdot)$ the above implies that $x(\theta^{n+1}, z) - \theta^{n+1} > x(\theta^n, z) - \theta^n$ which is a contradiction.

Proof of part (iii). Suppose that the maximum tax constraint (14) holds for $\theta = \theta^1$. Then given that the truth-telling constraint (15) also holds, from part (i), (14) holds $\forall\theta$.

Proof of part (iv). Condition (15) for $\theta = \theta^n$ implies that

$$v(x(\theta^n, z)) + \beta V^+(\theta^n, z) \geq v(x(\theta^1, z) + \theta^n - \theta^1) + \beta V^+(\theta^1, z)$$

which when combined with the limited commitment constraint (16) for $\theta = \theta^1$ implies that

$$v(x(\theta^n, z)) + \beta V^+(\theta^n, z) \geq v(x(\theta^1, z) + \theta^n - \theta^1) - v(x(\theta^1, z)) + v(f(i(z)) + \theta^1) + \beta \underline{V}. \tag{Appendix B.4}$$

The left hand side of ($AppendixB.4$) equals the left hand side of (16) for $\theta = \theta^n$. The concavity of $v(\cdot)$ implies that the right hand side of ($AppendixB.4$) weakly exceeds $v(f(i(z)) + \theta^n) + \beta\underline{V}$ since (14) implies that $x\left(\theta^1, z\right) \leq f(i) + \theta^1$.

Proof of part (v). Suppose that (14) is an equality for $\theta = \theta^1$. The fact that $V^+\left(\theta^1, z\right) \geq \underline{V}$ and together with (14) which is an equality implies that (16) is satisfied for $\theta = \theta^1$. Suppose that (16) binds for $\theta = \theta^1$. Since $V^+\left(\theta^1, z\right) \geq \underline{V}$, it follows that (14) is implied for $\theta = \theta^1$. ∎

The main takeaways from Lemma 7 are as follows. The solution to the program P_0 is the same as the solution to the relaxed problem which ignores the maximum tax constraint (14) and the limited commitment constraint (16) for $n > 1$ and which ignores the non-local truth-telling constraints (15). In addition, if one of either constraints (14) and (16) holds with equality, then the other is made redundant.

We now establish the existence of a solution to program P_0 with specific properties. Our first step is to show that there is always a solution in which the downward constraints in (15) bind.

Lemma 8. *If $V \in \left[V_0, \overline{V}\right)$, there exists a solution to program P_0 with the property that $C_{n+1,n} = 0 \ \forall n < N$ and $\forall z$.*

Proof. Consider a solution to the program α for which conditional on z, $C_{n+1} > C_n$ for some n. We can show that there exists a perturbation of this solution which satisfies all of the constraints and yields weakly greater welfare to the households and for which $C_{n+1,n} = 0$ for all n. Consider an alternative solution to the program $\widehat{\alpha}$ which is identical to α with the exception that $\widehat{V}^+(\theta, z)$ satisfies the following system of equations $\forall n < N$:

$$\textstyle\sum_{n=1}^{N} \pi(\theta^n)\, \widehat{V}^+(\theta^n, z) = \sum_{n=1}^{N} \pi(\theta^n)\, V^+(\theta^n, z) \qquad \text{(Appendix B.5)}$$

$$\widehat{V}^+(\theta^{n+1}, z) = \widehat{V}^+(\theta^n, z) + \left[v(x(\theta^n, z) + \theta^{n+1} - \theta^n) - v(x(\theta^{n+1}, z))\right]/\beta \qquad \text{(Appendix B.6)}$$

We now verify that the perturbed solution satisfies all of the constraints of the program. It satisfies the resource constraint (13) and the maximum tax constraint (14) since these are satisfied under the original allocation, and it satisfies the promise-keeping constraint (12) given (*AppendixB*.5) and the fact that (12) is also satisfied in the original allocation. From (*AppendixB*.6), it satisfies $C_{n+1,n} = 0 \ \forall n < N$. Moreover, it satisfies $C_{n,n+1} \geq 0 \ \forall n < N$ since if this were not the case, then together with the fact that $C_{n+1,n} = 0$, it would imply that

$$v(x(\theta^{n+1})) - v(x(\theta^{n+1}) - (\theta^{n+1} - \theta^n)) < v(x(\theta^n) + \theta^{n+1} - \theta^n) - v(x(\theta^n)),$$

which given the concavity of $v(\cdot)$ violates the fact that $x(\theta^n) \geq x\left(\theta^{n+1}\right) - \left(\theta^{n+1} - \theta^n\right)$ established in part (i) of Lemma 7. From part (ii) of Lemma 7, this implies that the truth-telling constraint (15) is satisfied for all θ and $\widehat{\theta}$. From part (iv) of Lemma 7, we need only verify the limited commitment constraint (16) for $\theta = \theta^1$, since (16) for other θ's are implied by the satisfaction of (14) and (15). This is implied by the fact that (*AppendixB*.5) and (*AppendixB*.6) imply that $\widehat{V}^+\left(\theta^1, z\right) \geq V^+\left(\theta^1, z\right)$. To see why this is true, note that the fact that $C_{n+1,n} \geq 0$ in the original solution implies that

$$\sum_{n=1}^{N} \pi(\theta^n)\, V^+(\theta^n, z) \geq$$

$$V^+(\theta^1, z) + \sum_{n=2}^{N} \pi(\theta^n) \sum_{l=1}^{n-1} (v(x(\theta^{n-l}, z) + \theta^{n-l+1} - \theta^{n-l}) - v(x(\theta^{n-l+1}, z)))/\beta$$

which combined with ($Appendix B.5$) and ($Appendix B.6$) implies that $\widehat{V}^{+}\left(\theta^{1}, z\right) \geq V^{+}\left(\theta^{1}, z\right)$. Analogous arguments imply that $\widehat{V}^{+}\left(\theta^{N}, z\right) \leq V^{+}\left(\theta^{N}, z\right)$, which together with part (i) of Lemma 7 implies that the feasibility constraint (18) is satisfied. Given ($Appendix B.5$) and ($Appendix B.6$) and the weak concavity of $J(\cdot)$, it follows that for all z,

$$\sum_{n=1}^{N} \pi\left(\theta^{n}\right) J(\widehat{V}^{+}\left(\theta^{n}, z\right)) \geq \sum_{n=1}^{N} \pi\left(\theta^{n}\right) J(V^{+}\left(\theta^{n}, z\right)), \tag{Appendix B.7}$$

since V^{+} is a mean preserving spread over $\widehat{V}^{+}$. Therefore, the household participation constraint (17) is satisfied. Therefore, $\widehat{\alpha}$ satisfies all of the constraints of the problem, and by ($Appendix B.7$), it weakly increases the welfare of the households. ■

What Lemma 8 shows in light of part (ii) of Lemma 7 is that there exists a solution to program P_0 for which the truth-telling constraint (15) is replaced with the constraint that $C_{n+1,n}=0$ and $x\left(\theta^{n}, z\right)-\theta^{n} \geq x\left(\theta^{n+1}, z\right)-\theta^{n+1}$ $\forall n<N$. Next we describe necessary properties of the solution to program P_0.

Lemma 9. *The solution to program P_0 has the following necessary properties:*

1. *If $V=\overline{V}$, then (17) binds $\forall z$,*
2. *If $V \in\left[V_0, \overline{V}\right)$, then (a) $i(z)>0$ $\forall z$, (b) $c(\theta, z)>0$ $\forall \theta, z$, and (c) (17) doesn't bind for some z.*

Proof. Proof of part (i). Suppose that $V=\overline{V}$ but that the household participation constraint (17) does not bind for some z. It is clear that conditional on z, the allocation α must provide a welfare of $\overline{V}$ to the policymaker since, otherwise it would be possible to make the policymaker strictly better by providing him the highest welfare for all z's and continuing to satisfy all of the constraints of the problem. Therefore, we can without loss of generality focus on the solution given $V=\overline{V}$ for which α is the same across z's. Moreover, by Lemma 8, we can consider such a solution for which $C_{n+1,n}=0$ $\forall n<N$. It is clear by the arguments of Lemma 8 that if (17) is slack under some original allocation for which $C_{n+1,n}>0$, then it continues to be slack under a perturbed allocation for which $C_{n+1,n}=0$. There are two cases to consider.

Case 1. Suppose it were the case that $C_{n-1,n}=0$ $\forall n \leq N$ so that $V^{+}\left(\theta^{n}, z\right)=V^{+}\left(\theta^{n-1}, z\right)$ and $c\left(\theta^{n}, z\right)=c\left(\theta^{n-1}, z\right)$ $\forall n<N$. Then this implies that $V^{+}\left(\theta^{n}, z\right)=\overline{V}$ and $c\left(\theta^{n}, z\right)=0$ $\forall n$. To see why, note that if $V^{+}\left(\theta^{n}, z\right)<\overline{V}$, then it would be possible to increase $V^{+}\left(\theta^{n}, z\right)$ by $\epsilon>0$ arbitrarily small $\forall n$ while continuing to satisfy all of the constraints of the problem and making the policymaker strictly better off. Suppose instead that $c\left(\theta^{n}, z\right)>0$. Then it would be possible to increase $i(z)$ by $\epsilon>0$, increase $x\left(\theta^{n}, z\right)$ by $f(i+\epsilon)-f(i)$ $\forall n$, and reduce $c\left(\theta^{n}, z\right)$ by ϵ $\forall n$ while continuing to satisfy the constraints of the problem and making the policymaker strictly better off. However, if it is the case that $V^{+}\left(\theta^{n}, z\right)=\overline{V}$ and $c\left(\theta^{n}, z\right)=0$ $\forall n$, then this implies that households are receiving a consumption of 0 forever, which violates (17).

Case 2. Suppose it is the case that $C_{n-1,n} > 0$ for some $n < N$. We rule out this case by induction. Suppose that $C_{N-1,N} > 0$. Then this implies that $V^+\left(\theta^N, z\right) = \overline{V}$ and $c\left(\theta^N, z\right) = 0$. This is because of analogous arguments as those of case 1. If $V^+\left(\theta^N, z\right) < \overline{V}$, then $V^+\left(\theta^N, z\right)$ can be increased by an arbitrarily small amount while continuing to satisfy all of the constraints of the problem and leaving the policymaker strictly better off. If instead $c\left(\theta^N, z\right) > 0$, then $x\left(\theta^N, z\right)$ can be increased by an arbitrarily small amount while continuing to satisfy all of the constraints of the problem and leaving the policymaker strictly better off. However, if $V^+\left(\theta^N, z\right) = \overline{V}$ and $c\left(\theta^N, z\right) = 0$, then part (i) of Lemma 7 implies that $c\left(\theta^n, z\right) = 0$ $\forall n$, which given that $C_{n+1,n} = 0$ $\forall n < N$ implies that $V^+\left(\theta^n, z\right) = \overline{V}$ $\forall n$. However, this contradicts the fact that $C_{N-1,N} > 0$. Therefore, $C_{N-1,N} = 0$.

Now suppose that $C_{\widetilde{n}-1,\widetilde{n}} = 0$ $\forall \widetilde{n} > n$ but that $C_{n-1,n} > 0$. Then this implies that $V^+\left(\theta^{\widetilde{n}}, z\right) = \overline{V}$ and $c\left(\theta^{\widetilde{n}}, z\right) = 0$ $\forall \widetilde{n} \geq n$, and this follows by analogous reasoning as in the case for which $C_{N-1,N} > 0$. However, as before, part (i) of Lemma 7 implies that $c\left(\theta^n, z\right) = 0$ $\forall n$, which given that $C_{n+1,n} = 0$ $\forall n$ implies that $V^+\left(\theta^n, z\right) = \overline{V}$ $\forall n$, contradicting the fact that $C_{n-1,n} > 0$.

Proof of part (ii.a). Suppose that $i\left(z\right) = 0$ for some z. By Lemma 8, we can consider such a solution for which $C_{n+1,n} = 0$ $\forall n < N$. We establish that this is not possible in the below steps.

Step 1. It must be the case then that the limited commitment constraint (16) is binding for some θ, since if this were not the case, one can perform the following perturbation to the solution α for the positive measure z for which $i\left(z\right) = 0$. Let $\widehat{i}\left(z, \epsilon\right) = \epsilon$ for some $\epsilon > 0$ arbitrarily small, and let $\widehat{c}\left(\theta^n, z, \epsilon\right) = c\left(\theta^n, \epsilon\right) + f\left(\epsilon\right) - \epsilon$, and leave the rest of the allocation unchanged. It can be easily verified that the perturbation satisfies the constraints of the program $(12) - (18)$. Moreover, it makes households strictly better off. Therefore (16) must bind with equality for some θ, and by part (iv) of Lemma 7, it must bind for $\theta = \theta^1$.

Step 2. This implies that $x\left(\theta^n, z\right) = \theta^n$ $\forall n$. To see why, consider a perturbation to the solution α which set $\widehat{i}\left(z, \epsilon\right) = \epsilon$ for some $\epsilon > 0$ arbitrarily small for all positive measure z for which $i\left(z\right) = 0$. Let $\widehat{x}\left(\theta^n, z, \epsilon\right)$ satisfy the following two equations

$$v(\widehat{x}(\theta^1, z, \epsilon)) - v(f(\widehat{i}\left(z, \epsilon\right) + \epsilon) + \theta^1) = v(x(\theta^1, z)) - v(f\left(i\left(z\right)\right) + \theta^1) \tag{Appendix B.8}$$

$$v\left(\widehat{x}\left(\theta^n, z, \epsilon\right)\right) - v(\widehat{x}(\theta^{n-1}, z, \epsilon) + \theta^n - \theta^{n-1}) = v\left(x\left(\theta^n, z\right)\right) - v(x(\theta^{n-1}, z) + \theta^n - \theta^{n-1}) \tag{Appendix B.9}$$

$\forall n > 1$. Finally, note that $\widehat{c}\left(\theta^n, z, \epsilon\right)$ is determined from the resource constraint

$$\widehat{c}\left(\theta^n, z, \epsilon\right) + \widehat{x}\left(\theta^n, z, \epsilon\right) = \omega - \widehat{i}\left(z, \epsilon\right) + f(\widehat{i}\left(z, \epsilon\right)) + \theta^n \ \forall n. \tag{Appendix B.10}$$

The rest of the allocation is left unchanged. It is straightforward to check that the perturbation satisfies $(13) - (18)$ so that it is an equilibrium and that it delivers a strictly higher continuation value to the incumbent. We can now show that it must make households strictly better off unless $x(\theta^n, z) = \theta^n \ \forall n$. Implicit differentiation of ($AppendixB.8$) – ($AppendixB.10$) around $\epsilon = 0$ implies that

$$\frac{d\widehat{c}(\theta^n, z, 0)}{d\epsilon} = f'(0) - 1 - \frac{d\widehat{x}(\theta^n, z, 0)}{d\epsilon} \quad \text{(Appendix B.11)}$$

$$\frac{d\widehat{x}(\theta^1, z, 0)}{d\epsilon} = \frac{v'(\theta^1)}{v'(x(\theta^1, z))} f'(0) \leq f'(0), \text{ and} \quad \text{(Appendix B.12)}$$

$$\frac{d\widehat{x}(\theta^n, z, 0)}{d\epsilon} = \frac{v'\left(x\left(\theta^{n-1}, z\right) + \theta^n - \theta^{n-1}\right)}{v'(x(\theta^n, z))} \frac{d\widehat{x}\left(\theta^{n-1}, z, 0\right)}{d\epsilon} \leq f'(0) \ \forall n > 1. \quad \text{(Appendix B.13)}$$

The final inequality in ($AppendixB.12$) is a strict inequality if $\theta^1 > x\left(\theta^1, z\right)$, and this holds by the concavity of $v(\cdot)$. Analogous arguments imply that the final inequality in ($AppendixB.13$) is a strict inequality if $\theta^1 > x\left(\theta^1, z\right)$ or if $x\left(\theta^{\widetilde{n}}, z\right) < x\left(\theta^{\widetilde{n}-1}, z\right) + \theta^{\widetilde{n}} - \theta^{\widetilde{n}-1}$ for any $1 < \widetilde{n} \leq n$. From the maximum tax constraint (14) and ($AppendixB.11$), it follows that the implied change in household consumption from an arbitrarily small increase in ϵ is positive if $x(\theta^n, z) < \theta^n$ for any n. Therefore, $x(\theta^n, z) = \theta^n \ \forall n$.

Step 3. It follows the fact that (16) is binding for some n and from the fact that $C_{n+1,n} = 0$ $\forall n < N$ that $V^+(\theta^n, z) = V^+\left(\theta^{n-1}, z\right) = \underline{V}$. However, one can show that this is suboptimal. From the proof of Lemma 6, it is clear that $V_0 > \underline{V}$. Consider then a perturbation to the solution α which set $\widehat{i}(z, \epsilon) = \epsilon$ for some $\epsilon > 0$ arbitrarily small for all positive measure z for which $i(z) = 0$. Moreover, let $\widehat{c}(\theta^n, z, \epsilon)$ is determined from

$$\widehat{c}(\theta^n, z, \epsilon) + x(\theta^n, z) = \omega - \widehat{i}(z, \epsilon) + f(\widehat{i}(z, \epsilon)) + \theta^n \ \forall n,$$

so that it is clear that consumption increases from the perturbation. Finally, let

$$\widehat{V}^+(\theta^n, z, \epsilon) = \widehat{V}^+(\theta^{n-1}, z, \epsilon) = (v(f(\widehat{i}(z, \epsilon)) + \theta^N) - v(\theta^N))/\beta + \underline{V}.$$

It can easily be verified that the perturbation satisfies (12) – (18). Moreover, for ϵ sufficiently low, the value of $J(\widehat{V}^+(\theta^n, z, \epsilon)) - J(\underline{V}) = 0$, where this follows from part (ii) of Lemma 5. Therefore, the perturbation makes households strictly better off.

Proof of part (ii.b). Suppose that $c(\theta^n, z) = 0$ for some n. In order to rule out this case, we take the following approach. We establish that there exists a perturbed allocation which gives some welfare $V + \epsilon$ for $\epsilon \gtreqless 0$ small to the policymaker which makes households infinitely better off on the margin. This leads to a contradiction because if $\epsilon > 0$, this implies that $J(V)$ is upward sloping, violating Lemma 5, and if instead $\epsilon < 0$, this implies that $J(V)$ has a slope of $-\infty$, implying that $V = \overline{V}$, where this follows from the concavity of $J(V)$ established in Lemma 5. By Lemma 8, we can perturb around a solution for which $C_{n+1,n} = 0 \ \forall n < N$. By part (i) of Lemma 7 and the resource constraint (13), it follows that if $c(\theta^n, z) = 0$ for some n, then $c\left(\theta^1, z\right) = 0$ and there exists some n^* such that $c(\theta^n, z) = 0$ for all $n \leq n^*$. There are several cases to consider.

Case 1. Suppose that $x(\theta^n, z) > 0\ \forall n$ and $n^* = N$ so that $c(\theta^n, z) = 0\ \forall n$. This implies that $C_{n,n+1} = 0\ \forall n < N$, so that $V^+\left(\theta^{n+1}, z\right) = V^+(\theta^n, z)$. Note that it cannot be that $V^+(\theta^n, z) = \overline{V}$, since from part (i), this would imply that household welfare conditional on z is $u(0) + \beta u(\omega)/(1-\beta)$, violating (17). It therefore follows given the concavity of $J(V)$ that the slope of $J(V^+(\theta^n, z))$ to the right of $V^+(\theta^n, z)$ is well defined and bounded away from $-\infty$. Consider the following perturbation. Let $\widehat{c}(\theta^n, z, \epsilon) = \epsilon$ and $\widehat{x}(\theta^n, z, \epsilon) = x(\theta^n, z) - \epsilon$. Moreover, let

$$\widehat{V}^+(\theta^n, z) = \sum_{n=1}^{N} \pi(\theta^n) \frac{v(x(\theta^n, z, \epsilon)) - v(\widehat{x}(\theta^n, z, \epsilon))}{\beta} + V^+(\theta^n, z).$$

Leave the rest of the allocation unchanged. It is straightforward to see that (13) − (18) is satisfied so that the perturbed allocation is an equilibrium. Moreover, it follows by Inada conditions and the fact that the slope of $J(V^+(\theta^n, z))$ to the right of $V^+(\theta^n, z)$ is well defined so that by the Inada conditions on $u(\cdot)$ rate of increase in the welfare of the households is ∞ for arbitrarily small ϵ.

Case 2. Suppose that $x(\theta^n, z) > 0\ \forall n$ and $n^* < N$. Note that in this case, it must be that $V^+(\theta^n, z) < \overline{V}$ for $n \leq n^*$, since if this were not the case, then $V^+(\theta^n, z) = \overline{V}\ \forall n$ which given that $C_{n+1,n} = 0$ implies that $c(\theta^n, z) = 0\ \forall n$, leading to a contradiction. Consider the following perturbation. Let $\widehat{x}(\theta^n, z, \epsilon) = x(\theta^n, z) - \epsilon\ \forall n \leq n^*$. Moreover, for $n \leq n^*$, let

$$\widehat{V}^+(\theta^n, z) = \frac{1}{\sum_{n=1}^{n^*} \pi(\theta^n)} \left(\sum_{n=1}^{n^*} \pi(\theta^n) \frac{v(x(\theta^n, z, \epsilon)) - v(\widehat{x}(\theta^n, z, \epsilon))}{\beta} \right) + V^+(\theta^n, z).$$

For $n = n^* + 1$, let $\widehat{x}(\theta^n, z, \epsilon)$ satisfy

$$\begin{aligned} v(\widehat{x}(\theta^n, z, \epsilon)) - v(\widehat{x}(\theta^{n-1}, z, \epsilon) + \theta^n - \theta^{n-1}) - \beta \widehat{V}^+(\theta^{n-1}, z) &= \\ v(x(\theta^n, z)) - v(x(\theta^{n-1}, z) + \theta^n - \theta^{n-1}) - \beta V^+(\theta^{n-1}, z)& \end{aligned}$$

and for $n > n^* + 1$, let $\widehat{x}(\theta^n, z, \epsilon)$ satisfy

$$v(\widehat{x}(\theta^n, z, \epsilon)) - v(\widehat{x}(\theta^{n-1}, z, \epsilon) + \theta^n - \theta^{n-1}) = v(x(\theta^n, z)) - v(x(\theta^{n-1}, z) + \theta^n - \theta^{n-1}).$$

Define $\widehat{c}(\theta^n, z, \epsilon) = c(\theta^n, z) - (\widehat{x}(\theta^n, z, \epsilon) - x(\theta^n, z))$. Note that concavity implies that $\widehat{x}(\theta^n, z, \epsilon) > x(\theta^n, z)$ for $n > n^*$. Leave the rest of the allocation unchanged. It is straightforward to verify that (13) − (18) are satisfied so that the perturbed allocation is an equilibrium. The only issue to note in establishing this is that (14) cannot hold with equality for $n > n^*$ in the original allocation so that by continuity it is satisfied in the perturbed allocation. Note that if it were the case that (14) held with equality, then this would imply by part (iii) of Lemma 7 that (14) holds with equality for $n = 1$. But if that were true, $c(\theta^n, z) = 0$ for $n > n^*$ leading to a contradiction. Analogous arguments then as those of case 1 imply that the rate of increase in the welfare of the households is ∞ for arbitrarily small ϵ.

Case 3. Suppose that $x(\theta^n, z) = 0$ for some n. Note that if it is the case that $x(\theta^n, z) > 0$ for all n for which $c(\theta^n, z) = 0$, then the arguments case 2 can be utilized, since the same perturbation is feasible in this case. We are left then to consider the case for which $c(\theta^n, z) = x(\theta^n, z) = 0$ for some n. We can establish that this can only be true for $n = 1$. To see why, note that if $c(\theta^n, z) = 0$ for any n, then $c(\theta^1, z) = 0$ by part (i) of Lemma 7. Moreover, (13) implies that for all n for which $c(\theta^n, z) = 0$, $x(\theta^n, z)$ is a strictly increasing function of n. Therefore, we are left to consider the case for which $c(\theta^1, z) = x(\theta^1, z) = 0$ and $x(\theta^n, z) > 0$ if $c(\theta^n, z) = 0$ for $n > 1$. From (13), in order that $x(\theta^n, z) = 0$ and $c(\theta^n, z) = 0$, it must be that $i(z) > i^*$ so that $\omega - i(z) + f(i(z)) + \theta^1 = 0$. Now consider the following perturbation to α. Let $\widehat{i}(z, \epsilon) = i(z) - \epsilon$ for $\epsilon > 0$ arbitrarily small and let

$$\widehat{c}(\theta^n, z, \epsilon) = c(\theta^n, z) + (f(\widehat{i}(z, \epsilon)) - \widehat{i}(z, \epsilon)) - (f(i(z)) - i(z)).$$

Leave the rest of the allocation as it is. It is straightforward to see that since (14) is a strict inequality $\forall n$ in the original allocation, (12) − (18) are satisfied in the perturbed allocation by continuity for sufficiently small ϵ. Since $\widehat{c}(\theta^n, z, \epsilon) > c(\theta^n, z)$ it follows that household welfare is increased under the perturbation, implying that the original allocation was suboptimal.

Proof of part (ii.c). Suppose that (17) binds $\forall z$. This implies that $J(V) = u(\omega)/(1-\beta)$. Given that $J(V) \geq u(\omega)/(1-\beta)$ for all V, this implies given the fact that $J(V)$ is weakly concave and weakly decreasing that $J(V) = u(\omega)/(1-\beta)$ for all V. However, one can show that this is not true by constructing an allocation which provides households a welfare which strictly exceeds $u(\omega)/(1-\beta)$. Construct an equilibrium as in the proof of Lemma 2 with the exception that $c_t(q_t, z_t, \theta_t) = \omega + \epsilon$, and $x_t(q_t, z_t, \theta_t) = f(i^*) - i^* + \theta_t - \epsilon$ for all θ_t for some $\epsilon > 0$ sufficiently small. By the arguments in the proof of Lemma 2, the allocation satisfies all the equilibrium constraints. Moreover, it provides households with a welfare which strictly exceeds $u(\omega)/(1-\beta)$, violating the fact that $J(V) = u(\omega)/(1-\beta)$ for all V. ■

Lemma 10. *$J(V)$ is continuously differentiable in V for $V \in (\underline{V}, \overline{V})$.*

Proof. We use Lemma 1 of Benveniste and Scheinkman (1979) to prove the continuous differentiability of $J(V)$ for $V \geq V_0$. In particular we show that if there exists a function $Q(V+\epsilon)$ for $\epsilon \gtreqless 0$ which is differentiable, weakly concave, and satisfies

$$Q(V+\epsilon) \leq J(V+\epsilon) \qquad \text{(Appendix B.14)}$$

for arbitrarily small values $|\epsilon|$ where $(Appendix B.14)$ is an equality if $\epsilon = 0$. By Lemma 1 of Benveniste and Scheinkman (1979) then $J(V)$ is continuously differentiable at V. To do so we first characterize a potential solution α conditional on $V \in [V_0, \overline{V})$, construct a perturbed solution $\widehat{\alpha}(\epsilon)$, and then check that this perturbed solution satisfies constraints (13) − (18) for $\epsilon \gtreqless 0$ arbitrarily small and provides $V + \epsilon$ to the policymaker. From Lemmas 6, 8, and 9, we can perturb around an original solution α with the following properties for some positive measure z:

$P(z) = 1$, $C_{n+1,n} = 0$ for all $n < N$, $i(z) > 0$, $c(\theta, z) > 0\ \forall\theta$, and the household participation constraint (17) does not bind. We let $\widetilde{Z}$ correspond to such z and let $q_{\widetilde{Z}} = \Pr(z \in \widetilde{Z})$.

Case 1. Suppose that $x(\theta, z) > 0\ \forall\theta$ and $\forall z \in \widetilde{Z}$. Define $\widehat{\alpha}(\epsilon)$ as follows. If $z \notin \widetilde{Z}$, then the element of $\widehat{\alpha}(\epsilon)$ is identical to the element of α. If instead $z \in \widetilde{Z}$, then let $\widehat{\alpha}(\epsilon)$ be identical to α, with the exception that $\forall n$,

$$\begin{aligned}\widehat{i}(z,\epsilon) &= i(z,\epsilon) + \xi^i(z,\epsilon), \\ \widehat{c}(\theta^n, z, \epsilon) &= c(\theta^n, z, \epsilon) + \xi^c(\theta^n, z, \epsilon), \text{ and} \\ \widehat{x}(\theta^n, z, \epsilon) &= x(\theta^n, z, \epsilon) + \xi^x(\theta^n, z, \epsilon)\end{aligned}$$

for $\xi(z,\epsilon) = \left\{\xi^i(z,\epsilon), \{\xi^c(\theta^n, z, \epsilon), \xi^x(\theta^n, z, \epsilon),\}_{n=1}^N\right\}$ which satisfy

$$\sum_{n=1}^{N} \pi(\theta^n)\, v(x(\theta^n, z)) + \epsilon/q_{\widetilde{Z}} = \sum_{n=1}^{N} \pi(\theta^n)\, v(\widehat{x}(\theta^n, z, \epsilon)) \tag{Appendix B.15}$$

$$\widehat{c}(\theta^n, z, \epsilon) + \widehat{x}(\theta^n, z, \epsilon) = \omega - \widehat{i}(z,\epsilon) + f(\widehat{i}(z,\epsilon)) + \theta^n\ \forall n \tag{Appendix B.16}$$

$$v(\widehat{x}(\theta^{n+1}, z, \epsilon)) - v(\widehat{x}(\theta^n, z, \epsilon) + \theta^{n+1} - \theta^n) = v(x(\theta^{n+1}, z)) - v(x(\theta^n, z) + \theta^{n+1} - \theta^n), \forall n < N \tag{Appendix B.17}$$

$$v(\widehat{x}(\theta^1, z, \epsilon)) - v(f(\widehat{i}(z,\epsilon)) + \theta^1) = v(x(\theta^1, z)) - v(f(i(z)) + \theta^1) \tag{Appendix B.18}$$

for a given ϵ. $(Appendix B.15) - (Appendix B.18)$ corresponds to $2N+1$ equations and $2N+1$ unknowns, where the ξ's all equal 0 if ϵ equals 0. Since $u(\cdot)$ and $v(\cdot)$ are continuously differentiable, then every element of $\xi(z,\epsilon)$ for a given z is a continuously differentiable function of ϵ around $\epsilon = 0$. Note that it it straightforward to see from $(Appendix B.15) - (Appendix B.18)$ that $\xi^i(z,\epsilon)$ and $\xi^x(\theta, z, \epsilon)$ rise in ϵ. The change in $\xi^c(\theta, z, \epsilon)$ is ambiguous. Define $Q(V+\epsilon)$ as the household welfare implied by the perturbed allocation $\widehat{\alpha}(\epsilon)$. It follows that $Q(V) = J(V)$ and that $Q(V+\epsilon)$ is continuously differentiable around $\epsilon = 0$. Note that because $J(V)$ is concave with $Q(V) = J(V)$, satisfaction of $(Appendix B.14)$ implies that $Q(V)$ is also locally concave around V.

We are left to verify that $(Appendix B.14)$ is satisfied. In order to do this, verify that every element of $\widehat{\alpha}(\epsilon)$ satisfies $(13) - (18)$ for a given promised value $V + \epsilon$ since this implies that $\widehat{\alpha}(\epsilon)$ is a potential solution to the program so that $(Appendix B.14)$ must hold. Satisfaction of the promise-keeping constraint (12) under α together with $(Appendix B.15)$ implies that $\widehat{\alpha}(\epsilon)$ satisfies (12) for continuation value $V + \epsilon$. Satisfaction of $(Appendix B.16)$ implies that $\widehat{\alpha}(\epsilon)$ satisfies satisfaction of the resource constraint (13). Now let us check that the truth-telling constraint (15) is satisfied. To do this, we appeal to part (ii) of Lemma 7 and simply check that $C_{n+1,n} \geq 0$ and $C_{n,n+1} \geq 0$ under $\widehat{\alpha}(\epsilon)$. Given that $C_{n+1,n} = 0$ under α, $(Appendix B.17)$ guarantees that $C_{n+1,n} = 0$ under $\widehat{\alpha}(\epsilon)$. Note furthermore that if $C_{n,n+1} > 0$ under α, then $C_{n,n+1} > 0$ under $\widehat{\alpha}(\epsilon)$ for sufficiently small ϵ by continuity. We are left to consider the situation for which $C_{n,n+1} = 0$ under α. In this case, $x\left(\theta^{n+1}, z\right) = x(\theta^n, z) + \theta^{n+1} - \theta^n$, which from

($AppendixB.17$) means that $\widehat{x}\left(\theta^{n+1}, z, \epsilon\right) = \widehat{x}\left(\theta^{n}, z, \epsilon\right) + \theta^{n+1} - \theta^{n}$ so that $C_{n,n+1} = 0$ under $\widehat{\alpha}\left(\epsilon\right)$ as well. Thus, (15) is satisfied under $\widehat{\alpha}\left(\epsilon\right)$. Now let us check that the maximum tax constraint (14) is satisfied under $\widehat{\alpha}\left(\epsilon\right)$. By part (iii) of Lemma 7, it is sufficient to check that this is the case if $\theta = \theta^1$ since (15) is satisfied. This is guaranteed by the fact that (14) is satisfied under α and by ($AppendixB.18$). We now check the limited commitment constraint (16). By part (iv) of Lemma 7, it is sufficient to check that this is the case if $\theta = \theta^1$ since (14) and (15) are satisfied. This is also guaranteed by the fact that (16) is satisfied under α and by ($AppendixB.18$). Since (17) is slack under α, then it is also slack under $\widehat{\alpha}\left(\epsilon\right)$ for sufficiently small ϵ. Finally since (18) is satisfied under α, it is also satisfied under $\widehat{\alpha}\left(\epsilon\right)$. Since the perturbed allocation satisfies all of the constraints, it follows that ($AppendixB.14$) holds.

Case 2. Suppose now that it is the case that $x\left(\theta, z\right) = 0$ for some positive measure z, and with some abuse of notation, relabel $\widetilde{Z}$ as the set of all such z's. We will prove that in this case, $J'\left(V\right) = 0$. Define $\widehat{\alpha}\left(\epsilon\right)$ for $\epsilon > 0$ analogously to case 1, where this is feasible since $\widehat{x}\left(\theta^n, z, \epsilon\right) > x\left(\theta^n, z\right)$ $\forall n$. It follows by implicit differentiation given the Inada conditions on $v\left(\cdot\right)$ and the fact that $\widehat{i}\left(z, \epsilon\right)$ and $\widehat{c}\left(\theta^n, z, \epsilon\right)$ are interior that

$$\lim_{\epsilon>0,\epsilon\to0} \frac{Q\left(V+\epsilon\right) - Q\left(V\right)}{\epsilon} = 0 \leq \lim_{\epsilon>0,\epsilon\to0} \frac{J\left(V+\epsilon\right) - J\left(V\right)}{\epsilon}, \qquad \text{(Appendix B.19)}$$

where we have used the fact that $Q\left(V+\epsilon\right) \leq J\left(V+\epsilon\right)$ for $\epsilon > 0$ and $Q\left(V\right) = J\left(V\right)$. Given that $J\left(V\ \right)$ is weakly decreasing, it follows that the last weak inequality in ($AppendixB.19$) binds with equality. Since $J\left(V\right)$ is weakly decreasing and weakly concave, it follows that

$$0 \geq \lim_{\epsilon>0,\epsilon\to0} \frac{J\left(V\right) - J\left(V-\epsilon\right)}{\epsilon} \geq \lim_{\epsilon>0,\epsilon\to0} \frac{J\left(V+\epsilon\right) - J\left(V\right)}{\epsilon} = 0,$$

which implies that $J\left(V\right)$ is differentiable at V with $J'\left(V\right) = 0$. ■

Appendix C. Proofs of Section 4

Appendix C.1. Proof of Lemma 3

We consider the solution to the program P_0, which ignores the truth-telling constraint (15) so that there is no private information. We first prove the following preliminary result.

Lemma 11. *The solution to program P_0, which ignores constraint* (15) *has the following features:*

1. *If $V = \overline{V}$, then $c\left(\theta^n, z\right) = c^{\max}\left(\theta^n\right)$, $x\left(\theta^n, z\right) = x^{\max}\left(\theta^n\right)$, $i\left(z\right) = i^*$, and $V^+\left(\theta^n, z\right) = \overline{V}$ $\forall z$,*
2. *If $V < \overline{V}$, but* (17) *binds conditional on z, then $c\left(\theta^n, z\right) = c^{\max}\left(\theta^n\right)$, $x\left(\theta^n, z\right) = x^{\max}\left(\theta^n\right)$, $i\left(z\right) = i^*$, and $V^+\left(\theta^n, z\right) = \overline{V}$.*

Proof. Proof of part (i). Let

$$\overline{V}^* = \max_{\{c_t, x_t, i_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t v(x_t) \right\}$$

$$\text{s.t.}$$

$$c_t + x_t = \omega - i_t + f(i_t) + \theta_t \ \forall t,$$

$$\text{and } E_t \left\{ \sum_{k=0}^{\infty} \beta^k u(c_{t+k}) \right\} \geq u(\omega) / (1-\beta) \ \forall t.$$

If it is the case that $\overline{V} = \overline{V}^*$, then starting from $V = \overline{V}$, the unique solution to program P_0, which ignores constraint (15) coincides with the unique solution to the above program. Note that such a solution satisfies the features described in the statement of part (i) of the lemma.

We now verify that $\overline{V} = \overline{V}^*$. To do this, we consider the allocation described in the statement of the lemma and verify that it satisfies all of the equilibrium constraints. From (21), the maximum tax constraint (14) is satisfied $\forall$ n, where we have used the fact that $c^{\max}(\theta^n)$ is increasing in θ^n. The resource constraint (13) and the household participation constraint (17) are also clearly satisfied, so we are left to consider the limited commitment constraint (16), and this is guaranteed by (21). Therefore, the solution satisfies the equilibrium constraints so that $\overline{V} = \overline{V}^*$.

Proof of part (ii). Now suppose that given V, (17) binds conditional on z so that households receive a continuation welfare of $u(\omega)/(1-\beta)$. Optimality then requires that the policymaker receive a continuation value of $\overline{V}$ conditional on z, since otherwise it would be possible to make the policymaker strictly better off while leaving the households as well off, violating the fact that $J(V)$ is downward sloping. By part (i), there is a unique method of providing this continuation value described in the statement of the lemma. ■

We now proceed in proving Lemma 3.

Proof of part (i) Let λ, $\pi(\theta^n)\,\nu(\theta^n, z)\,dz$, $\pi(\theta^n)\,\kappa(\theta^n, z)\,dz$, $\pi(\theta^n)\psi(\theta^n, z)\,dz$, and $\eta(z)\,dz$ correspond to the Lagrange multipliers on the promise-keeping constraint (12), the resource constraint (13), the maximum tax constraint (14), the limited commitment constraint (16), and the household participation constraint (17), respectively. Let $\beta\pi(\theta^n)\,\underline{\mu}(\theta^n, z)\,dz$ also $\beta\pi(\theta^n)\,\overline{\mu}(\theta^n, z)\,dz$, and $\pi(\theta^n)\,\zeta(\theta^n, z)\,dz$ correspond to the Lagrange multipliers on the constraints that $V'(\theta^n, z) \geq \underline{V}$, $V^+(\theta^n, z) \leq \overline{V}$, and $x(\theta^n, z) \geq 0$, respectively. Analogous arguments as those of Lemma 9 in Appendix Appendix B imply that the constraint that $i(z) \geq 0$ and $c(\theta, z) \geq 0$ is always slack and can therefore be ignored. First order conditions imply:

$$u'\left(c\left(\theta^n,z\right)\right)\left(P(z)+\eta\left(z\right)\right)=\nu\left(\theta^n,z\right), \tag{Appendix C.1}$$

$$\left(P(z)\lambda+\psi\left(\theta^n,z\right)\right)v'\left(x\left(\theta^n,z\right)\right)=\nu\left(\theta^n,z\right)+\kappa\left(\theta^n,z\right)-\zeta\left(\theta^n,z\right) \tag{Appendix C.2}$$

$$f'\left(i\left(z\right)\right)-1=\frac{\sum_{n=1}^{N}\pi\left(\theta^n\right)\left(-\kappa\left(\theta^n,z\right)f'\left(i\left(z\right)\right)+\psi\left(\theta^n,z\right)v'\left(f\left(i\left(z\right)\right)+\theta^n\right)f'\left(i\left(z\right)\right)\right)}{\sum_{n=1}^{N}\pi\left(\theta\right)\nu\left(\theta^n,z\right)} \tag{Appendix C.3}$$

$$J'\left(V^+\left(\theta^n,z\right)\right)\left(P(z)+\eta\left(z\right)\right)=\left\{-P(z)\lambda-\psi\left(\theta^n,z\right)-\underline{\mu}\left(\theta^n,z\right)+\overline{\mu}\left(\theta^n,z\right)\right\} \tag{Appendix C.4}$$

and the Envelope condition yields:

$$J'\left(V\right)=-\lambda. \tag{Appendix C.5}$$

To prove that a distortion emerges, we consider the situation with $V=V_0$ which occurs at $t=0$. Analogous argument as those of part (ii) of Lemma 5 in Appendix Appendix B imply that $J'\left(V_0\right)=0$, hence at V_0, $\lambda=0$. Moreover, analogous arguments as those of Lemma 6 in Appendix Appendix B imply that $P\left(z\right)=1$ $\forall z$ if $V\geq V_0$. Finally, note that if $V=V_0$, then necessarily $\eta\left(z\right)=0$. If not, then (17) binds for some z and households would be receiving a continuation welfare of $u\left(\omega\right)/\left(1-\beta\right)<\overline{J}$ with positive probability, violating the fact that $J\left(V_0\right)=\overline{J}$ by definition. Now suppose by contradiction that $i\left(z\right)\geq i^*$ starting for $V=V_0$. We establish that this is not possible in two steps.

Step 1. It is not possible that $\psi\left(\theta^n,z\right)=0$ $\forall n$. Suppose this were the case. Then (*AppendixC*.2) and (*AppendixC*.3) given that $\lambda=0$ imply that $\zeta\left(\theta^n,z\right)>0$ so that $x\left(\theta^n,z\right)=0$ $\forall n$. Moreover, (*AppendixC*.4) given that $\lambda=\eta\left(z\right)=0$ implies that $J'\left(V^+\left(\theta^n,z\right)\right)=0$.[37] Since V_0 corresponds to the highest continuation value V which the policymaker can receive subject to $J'\left(V\right)=0$, it follows then that for all θ^n and z: $V^+\left(\theta^n,z\right)\leq V_0$, so that $V_0\leq v\left(0\right)/\left(1-\beta\right)$. However, if this is the case, together with Assumption 1 it implies that (16) is violated, leading to a contradiction.

Step 2. Consider for some n, $\psi\left(\theta^n,z\right)>0$ so that (16) binds for such n. The same arguments as those of part (v) of Lemma 7 in Appendix Appendix B imply that constraint (14) can be ignored in this case so that $\kappa\left(\theta^n,z\right)=0$ for such n. Moreover, note that for all other n for which $\psi\left(\theta^n,z\right)=0$, $x\left(\theta^n,z\right)=0$ by the same reasoning as in step 1, and it trivially follows that (14) cannot bind for such n either. Therefore, $\kappa\left(\theta^n,z\right)=0$ $\forall n$. However, if this is the case, then (*AppendixC*.3) implies that $i\left(z\right)<i^*$, leading to a contradiction.

Proof of part (ii) To prove this second result, we first establish that $J'\left(V^+\left(\theta^n,z\right)\right)\leq J'\left(V\right)$ $\forall z$ and $\forall n$. Suppose first that $\eta\left(z\right)>0$. This implies given Lemma 11 that $V^+\left(\theta^n,z\right)=\overline{V}$, and from the concavity of $J\left(V\right)$, this implies that $J'\left(V^+\left(\theta^n,z\right)\right)\leq J'\left(V\right)$. If instead $\eta\left(z\right)=0$, then

[37] This follows from the fact that since $J'\left(V^+\left(\theta^n,z\right)\right)$ is non-positive, $\overline{\mu}\left(\theta^n,z\right)=0$, and moreover, it is not possible that $\underline{\mu}\left(\theta^n,z\right)>0$, since from (*AppendixC*.4), this would lead to $J'\left(V^+\left(\theta^n,z\right)\right)<0$ and thus $V^+\left(\theta^n,z\right)>\underline{V}$, which is a contradiction with $\underline{\mu}\left(\theta^n,z\right)>0$.

(*AppendixC*.4) and the Envelope condition imply that if it is not the case that $V^+(\theta^n, z) = \overline{V}$, then

$$J'\left(V^+(\theta^n, z)\right) = J'(V) - \psi(\theta^n, z) \leq J'(V).$$

Therefore, given a stochastic sequence $\{V_t\}_{t=0}^{\infty}$, it must be that the associated stochastic sequence $\{J'(V_t)\}_{t=0}^{\infty}$ is monotonically decreasing. As such, from (*AppendixC*.5), this means that the sequence $\{\lambda_t\}_{t=0}^{\infty}$ is itself either converging or declining towards $-\infty$. This leaves us with two cases to consider.

Case 1. Suppose that λ_t converges to a finite number. From (*AppendixC*.4) and (*AppendixC*.5), this implies that $\{\psi_t\}_{t=0}^{\infty}$ is a sequence which converges to zero. Note that since V_t is bounded, there exists a convergent subsequence of $\{V_t\}_{t=0}^{\infty}$ which converges to some limiting continuation value V. By continuity, there is some limiting associated Lagrange multiplier $\psi = 0$. Given V, the first order condition for investment (*AppendixC*.3) implies that $i(z) \geq i^*$ in this limiting allocation. From Assumption 3, it follows that (14) cannot bind and therefore from (*AppendixC*.3), $i(z) = i^*$ in the limiting allocation.

Case 2. Suppose that λ_t diverges towards ∞ so that therefore $J'(V_t)$ diverges towards $-\infty$. In this case, it must be that $\{V_t\}_{t=0}^{\infty}$ converges towards $\overline{V}$, and this follows from the concavity of $J(\cdot)$. However, by the reasoning of Lemma 11, the limiting allocation here also entails $i(z) = i^*$.■

Appendix C.2. Proof of Lemma 4

We consider the solution to program P_0, which ignores the limited commitment constraint (16). Let λ, $\pi(\theta^n)\nu(\theta^n, z)\,dz$, $\pi(\theta^n)\kappa(\theta^n, z)\,dz$, and $\eta(z)\,dz$ correspond to the Lagrange multipliers on the promise-keeping constraint (12), the resource constraint (13), the maximum tax constraint (14), and the household participation constraint (17), respectively. Let $\beta\pi(\theta^n)\underline{\mu}(\theta^n, z)\,dz$, $\beta\pi(\theta^n)\overline{\mu}(\theta^n, z)\,dz$, and $\pi(\theta^n)\zeta(\theta^n, z)\,dz$ correspond to the Lagrange multipliers on the constraints that $V^+(\theta^n, z) \geq \underline{V}$, $V^+(\theta^n, z) \leq \overline{V}$, and $x(\theta^n, z) \geq 0$, respectively. Analogous arguments as those of Lemma 9 in Appendix Appendix B imply that the constraint that $i(z) \geq 0$ and $c(\theta, z) \geq 0$ is always slack and can therefore be ignored. Finally, by part (ii) of Lemma 7 in Appendix Appendix B, we need only consider the local constraints for (15). Let $\pi\left(\theta^{n+1}\right)\phi\left(\theta^{n+1}, \theta^n, z\right)dz$ and $\pi(\theta^n)\phi\left(\theta^n, \theta^{n+1}, z\right)dz$ correspond to the Lagrange multipliers on the downward and upward incentive compatibility constraint, respectively, where we define $\phi\left(\theta^n, \theta^{n-1}, z\right) = 0$ if $n = 1$ and $\phi\left(\theta^{n+1}, \theta^n, z\right) = 0$ if $n = N$. The first order condition with respect to $i(z)$ is:

$$f'(i(z)) - 1 = \frac{\sum_{n=1}^{N} \pi(\theta^n)\left(-\kappa(\theta^n, z) f'(i(z))\right)}{\sum_{n=1}^{N} \pi(\theta)\nu(\theta^n, z)}.$$

Therefore, if it is the case that $i(z) \neq i^*$, this implies that $i(z) > i^*$ and that (14) binds for some θ^n, contradicting Assumption 3. Therefore, $i(z) = i^*\ \forall z$.■

Appendix D. Proofs of Section 5

Appendix D.1. Proof of Proposition 2

Proof of part (i) To show that distortions emerge along the equilibrium path, we pursue the same strategy as in the proof of part (i) of Lemma 3 in Appendix Appendix C. Define all of the Lagrange multipliers as in the proofs of Lemma 3 and Lemma 4 in Appendix Appendix C. First order conditions yield:

$$u'\left(c\left(\theta^n,z\right)\right)\left(P(z)+\eta\left(z\right)\right)=\nu\left(\theta^n,z\right), \tag{Appendix D.1}$$

$$\left\{\begin{array}{c}\left(\begin{array}{c}P(z)\lambda+\phi\left(\theta^n,\theta^{n+1},z\right)\\+\phi\left(\theta^n,\theta^{n-1},z\right)+\psi\left(\theta^n,z\right)\end{array}\right)v'\left(x\left(\theta^n,z\right)\right)\\-\phi\left(\theta^{n-1},\theta^n,z\right)v'\left(x\left(\theta^n,z\right)+\theta^{n-1}-\theta^n\right)\frac{\pi(\theta^{n-1})}{\pi(\theta^n)}\\-\phi\left(\theta^{n+1},\theta^n,z\right)v'\left(x\left(\theta^n,z\right)+\theta^{n+1}-\theta^n\right)\frac{\pi(\theta^{n+1})}{\pi(\theta^n)}\end{array}\right\}=\nu\left(\theta^n,z\right)+\kappa\left(\theta^n,z\right)-\zeta\left(\theta^n,z\right) \tag{Appendix D.2}$$

$$f'\left(i\left(z\right)\right)-1=\frac{\sum_{n=1}^{N}\pi\left(\theta^n\right)\left(-\kappa\left(\theta^n,z\right)f'\left(i\left(z\right)\right)+\psi\left(\theta^n,z\right)v'\left(f\left(i\left(z\right)\right)+\theta^n\right)f'\left(i\left(z\right)\right)\right)}{\sum_{n=1}^{N}\pi\left(\theta\right)\nu\left(\theta^n,z\right)} \tag{Appendix D.3}$$

$$J'\left(V^+\left(\theta^n,z\right)\right)\left(P(z)+\eta\left(z\right)\right)=\left\{\begin{array}{c}-P(z)\lambda-\phi\left(\theta^n,\theta^{n-1},z\right)-\phi\left(\theta^n,\theta^{n+1},z\right)\\+\phi\left(\theta^{n-1},\theta^n,z\right)\frac{\pi(\theta^{n-1})}{\pi(\theta^n)}+\phi\left(\theta^{n+1},\theta^n,z\right)\frac{\pi(\theta^{n+1})}{\pi(\theta^n)}\\-\psi\left(\theta^n,z\right)-\underline{\mu}\left(\theta^n,z\right)+\overline{\mu}\left(\theta^n,z\right)\end{array}\right\} \tag{Appendix D.4}$$

and the Envelope condition yields:

$$J'\left(V\right)=-\lambda. \tag{Appendix D.5}$$

The following lemma, which is implied by Assumption 3, is useful for the remainder of our analysis.

Lemma 12. *Suppose that $i\left(z\right)\neq i^*$. Then (16) holds as an equality for $\theta^n=\theta^1$ with $i\left(z\right)<i^*$.*

Proof. Suppose that $i\left(z\right)\neq i^*$ and suppose first that $i\left(z\right)>i^*$. Given (*AppendixD*.3), this is only possible if the maximum tax constraint (14) binds for some n contradicting Assumption 3. Therefore, it must be that if $i\left(z\right)\neq i^*$, then $i\left(z\right)<i^*$. Now suppose that $i\left(z\right)<i^*$ but that the limited commitment constraint (16) holds as a strict inequality for $\theta^n=\theta^1$. Then from part (iv) of Lemma 7 in Appendix Appendix B, (16) is a strict inequality for all θ^n, which from (*AppendixD*.3) implies that $i\left(z\right)\geq i^*$, which is a contradiction. This establishes that (16) must hold with equality and that $i\left(z\right)<i^*$. ■

Now to prove our result, note as a reminder that from part (ii) of Lemma 5 in Appendix Appendix B, it is the case that $J'\left(V_0\right)=0$. Analogous arguments to those in the proof of Lemma 3 in Appendix Appendix C imply that $\eta\left(z\right)=0$. Now suppose that at V_0, $i\left(z\right)\geq i^*$, where from Lemma 12, this is only possible if $i(z) = i^*$. To show this is not the case, we proceed in two analogous steps to those in the proof of Lemma 3 in Appendix Appendix C.

Step 1. It is not possible that $\psi(\theta^n, z) = 0 \; \forall n$. Suppose this was the case. From (*AppendixD*.4) and (*AppendixD*.5), this implies that given z,

$$\sum_{n=1}^{N} \pi(\theta^n) J'\left(V^+(\theta^n, z)\right) = J'(V_0) - \sum_{n=1}^{N} \pi(\theta^n) \overline{\mu}(\theta^n, z) + \sum_{n=1}^{N} \pi(\theta^n) \overline{\mu}(\theta^n, z). \quad \text{(Appendix D.6)}$$

Given that $J(\cdot)$ is weakly decreasing it follows that the left hand side of equation (Appendix D.6) is weakly negative. This implies that $\forall n$, $\overline{\mu}(\theta^n, z) = 0$ and $J'(V^+(\theta^n, z)) = 0$. From part (ii) of Lemma 5 in Appendix Appendix B, this implies that $V^+(\theta^n, z) \leq V_0 \; \forall \, n$. Therefore, from (*AppendixD*.4) for $n = 1$ it must be that $\phi\left(\theta^1, \theta^2, z\right) = \phi\left(\theta^2, \theta^1, z\right) \frac{\pi(\theta^2)}{\pi(\theta^1)}$. This implies from (*AppendixD*.4) $\forall n$ that $\phi\left(\theta^n, \theta^{n+1}, z\right) = \phi\left(\theta^{n+1}, \theta^n, z\right) \frac{\pi(\theta^{n+1})}{\pi(\theta^n)}$ for all $n < N$. We now show that $\phi\left(\theta^n, \theta^{n-1}, z\right) = 0$ for all $n > 1$. Suppose not, then $\phi\left(\theta^N, \theta^{N-1}, z\right) > 0$. Since it must also be the case then that $\phi\left(\theta^{N-1}, \theta^N, z\right) > 0$, this means that

$$x(\theta^{N-1}) = x(\theta^N) - (\theta^N - \theta^{N-1}) < x(\theta^N). \quad \text{(Appendix D.7)}$$

Now consider (*AppendixD*.2) for $n = N$ given that $\phi\left(\theta^N, \theta^{N-1}, z\right) > 0$:

$$0 > \phi(\theta^N, \theta^{N-1}, z)[v'(x(\theta^N, z)) - v'(x(\theta^N, z) - (\theta^N - \theta^{N-1}))] = \nu(\theta^N, z) + \kappa(\theta^N, z) - \zeta(\theta^N, z). \quad \text{(Appendix D.8)}$$

From (*AppendixD*.1), it must be that $\nu(\theta^N, z) > 0$ and the maximum tax constraint (14) implies that $\kappa(\theta^N, z) \geq 0$, which means that for (*AppendixD*.8) to hold, it must be that $\zeta(\theta^N, z) > 0$ so that $x(\theta^N, z) = 0$. However, this implies from (*AppendixD*.7) that $x(\theta^{N-1}) < 0$, which is not possible. Therefore, $\phi(\theta^N, \theta^{N-1}, z) = 0$. Now suppose that $\phi(\theta^{N-1}, \theta^{N-2}, z) > 0$. Analogous reasoning to the above implies analogous conditions to (*AppendixD*.7) and (*AppendixD*.8) for $N-1$. But this leads to a contradiction since it implies that $x(\theta^{N-2}) < 0$, which is not possible. Similar arguments imply that $\phi\left(\theta^n, \theta^{n-1}, z\right) = 0$ for all $n > 1$.

Now consider (*AppendixD*.2) given that $\phi\left(\theta^n, \theta^{n-1}, z\right) = \phi\left(\theta^{n-1}, \theta^n, z\right) = 0$ for all $n > 1$. Since $\nu(\theta^n, z) > 0$ and $\kappa(\theta^n, z) \geq 0$, this means that $\zeta(\theta^n, z) > 0$ and $x(\theta^n, z) = 0$ for all n. Therefore, conditional on z and θ^n, the policymaker receives a continuation

$$v(0) + \beta V^+(\theta^n, z) \leq v(0) / (1 - \beta).$$

However, if that were true, then the limited commitment constraint (16) is violated by Assumption 1.

Step 2. We have established that $\psi(\theta^n, z) > 0$ for some θ^n. From part (iv) of Lemma 7 in Appendix Appendix B, this would only be the case for $n = 1$, so that $\psi\left(\theta^1, z\right) > 0$ and $\psi(\theta^n, z) = 0 \; \forall n > 1$. Moreover, from parts (iii) and (v) of Lemma 7 in Appendix Appendix B, constraint (14) is made redundant for all n and can be ignored since (16) binds with an equality, so that $\kappa(\theta^n, z) = 0 \; \forall n$. Given that $\nu(\theta^n, z) > 0$ from (*AppendixD*.1), this means that the right hand side of (*AppendixD*.3) is positive so that $i(z) < i^*$.■

Proof of part (ii) Suppose it were the case that $\Pr\{\lim_{t\to\infty} i_t = i^*\} > 0$. This would imply that there exist a stochastic sequence $\{i_t\}_{t=0}^{\infty}$ with positive measure which converges to i^*. Associated with such a stochastic sequence is a stochastic sequence $\{V_t\}_{t=0}^{\infty}$ which must includes within it at least one convergent stochastic subsequence, where this follows from the fact that V_t is bounded. Let Ψ correspond to the entire set of all limiting values of convergent stochastic subsequences of $\{V_t\}_{t=0}^{\infty}$ for every single stochastic sequence $\{i_t\}_{t=0}^{\infty}$ which converges to i^*. It follows by the continuity of the policy function that the solution to program P_0 starting from some $V \in \Psi$ yields a solution which admits $i(z) = i^* \; \forall z$. Moreover, it must be that $V^+(\theta, z) \in \Psi \; \forall \theta$ and $\forall z$ with positive measure since $V^+(\theta, z)$ must itself be within the set of limits of stochastic subsequences. Let $\underline{V}^+$ correspond to the infinum of continuation values in Ψ. It is clear that $\underline{V}^+ > V_0$, since if this were not the case, this would imply by the proof of part (i) given that $\lambda = 0$ by part (ii) of Lemma 5 in Appendix Appendix B that $i(z) \neq i^*$ for some $V \in \Psi$, leading to a contradiction. The strategy of our proof is to show that $\underline{V}'$ does not exist so as to create a contradiction. Before proceeding, we prove the following preliminary lemma.

Lemma 13. *Consider the solution to program P_0 given $V \in (V_0, \overline{V})$. Suppose that in the solution, (16) is slack $\forall \theta$ and $\forall z$. Then it must be that the solution admits $J'\left(V^+\left(\theta^1, z\right)\right) > J'(V) \; \forall z$.*

Proof. Note that by Lemma 6 in Appendix Appendix B, the solution must admit $P(z) = 1$ $\forall z$. There are two cases to consider.

Case 1. Suppose that the elements of α do not vary with z. Now suppose that $J'\left(V^+\left(\theta^1, z\right)\right) \leq J'(V) \; \forall z$. Part (i) of Lemma 7 in Appendix Appendix B together with the weak concavity of $J(V)$ then implies that $J'(V^+(\theta^n, z)) \leq J'(V) \; \forall n$ and $\forall z$, where we have used the fact that α does not depend on z. From Lemma 8 in Appendix Appendix B, one can perturb such a solution without changing households' welfare and continuing to satisfy the constraints of the problem by changing the values of $V^+(\theta^n, z)$ so that $C_{n+1,n} = 0$ for all $n < N$. Note that this perturbation weakly increases the value of $V^+\left(\theta^1, z\right)$ so that it remains the case that $J'\left(V^+\left(\theta^1, z\right)\right) \leq J'(V)$ $\forall z$. Such a solution corresponds to the solution to the following problem, where λ corresponds to the Lagrange multiplier on the promise-keeping (12):

$$J(V) = \max_{\alpha} \left\{ \int_0^1 \left[\begin{array}{c} \sum_{n=1}^{N} \pi(\theta^n) \left(u(c(\theta^n, z)) + \beta J(V^+(\theta^n, z))\right) \\ +\lambda \left(\sum_{n=1}^{N} \pi(\theta^n) \left(v(x(\theta^n, z)) + \beta V^+(\theta^n, z)\right) \right) \end{array} \right] dz \right\}$$

$$\text{s.t.,} \qquad c(\theta^n, z) + x(\theta^n, z) = \omega - i(z) + f(i(z)) + \theta^n \; \forall n, z, \qquad \text{(Appendix D.9)}$$

$$x(\theta^n, z) \leq f(i(z)) + \theta^n \; \forall n, z, \qquad \text{(Appendix D.10)}$$

$$x(\theta^{n+1},z) \leq x\left(\theta^{n},z\right)+\theta^{n+1}-\theta^{n} \ \forall n<N,z, \tag{Appendix D.11}$$

$$v(x(\theta^{n+1},z))+\beta V^{+}(\theta^{n+1},z)=v(x(\theta^{n},z)+\theta^{n+1}-\theta^{n})+\beta V^{+}\left(\theta^{n},z\right) \ \forall n<N,z, \tag{Appendix D.12}$$

$$\sum_{n=1}^{N}\pi\left(\theta^{n}\right)u\left(c\left(\theta^{n},z\right)+\beta J\left(V^{+}\left(\theta^{n},z\right)\right)\right) \geq u\left(\omega\right)/\left(1-\beta\right) \ \forall z, \tag{Appendix D.13}$$

$$\text{and } V^{+}\left(\theta^{n},z\right)\in\left[\underline{V},\overline{V}\right] \ \forall n,z. \tag{Appendix D.14}$$

The above program differs from the general program in the following fashion: It takes into account that replacement never occurs; it has removed constraints which do not bind; it has substituted (12) into the objective function taking into account that λ is the Lagrange multiplier on (12); and it has replaced the truth-telling constraint (15) with constraints (*AppendixD*.11) and (*AppendixD*.12) by using part (ii) of Lemma 7 and Lemma 8 in Appendix Appendix B.

Since $V>V_0$, we only need consider the case for which $\lambda>0$. This is because if $\lambda=0$, then $V\leq V_0$, where this follows from the Envelope condition in (*AppendixD*.5) and part (ii) of Lemma 5 in Appendix Appendix B. Define Lagrange multipliers analogously as in step 1. First order conditions with respect to $V^{+}\left(\theta^{n},z\right)$ yield:

$$J'\left(V^{+}\left(\theta^{n},z\right)\right)\left(1+\eta\left(z\right)\right)=-\lambda-\phi(\theta^{n},\theta^{n-1},z)+\phi(\theta^{n+1},\theta^{n},z)+\overline{\mu}\left(\theta^{n},z\right), \tag{Appendix D.15}$$

where we have taken into account that the fact that $V^{+}\left(\theta^{n},z\right)\geq V$ implies that $V^{+}\left(\theta^{n},z\right)>\underline{V}$ so that $\underline{\mu}\left(\theta^{n},z\right)=0$. Since the allocation does not depend on z, it is the case that $\eta\left(z\right)=0$, since otherwise the constraint (*AppendixD*.13) binds $\forall z$ so that $J\left(V\right)=u\left(\omega\right)/\left(1-\beta\right)$. The Envelope condition yields: $J'\left(V\right)=-\lambda$. From (*AppendixD*.15), since $J'\left(V^{+}\left(\theta^{n},z\right)\right)\leq J'\left(V\right)$ for $n=1$, this implies that $\phi\left(\theta^{2},\theta^{1},z\right)\leq 0$. For $n=2$, this implies that $\phi\left(\theta^{3},\theta^{2},z\right)\leq\phi\left(\theta^{2},\theta^{1},z\right)\leq 0$, and forward induction implies that $\phi\left(\theta^{N},\theta^{N-1},z\right)\leq\phi\left(\theta^{2},\theta^{1},z\right)\leq 0$. (*AppendixD*.15) for $n=N$ given that $J'\left(V^{+}\left(\theta^{n},z\right)\right)\leq J'\left(V\right)$ requires $\phi\left(\theta^{N},\theta^{N-1},z\right)\geq 0$, which thus implies that $\phi\left(\theta^{n+1},\theta^{n},z\right)=0 \ \forall n<N$. Therefore, constraint (*AppendixD*.12) can be ignored.

Let us assume and later verify that constraint (*AppendixD*.11) can also be ignored if constraint (*AppendixD*.12) can be ignored. First order conditions with respect to $c\left(\theta^{n},z\right)$ and $x\left(\theta^{n},z\right)$ together with (*AppendixD*.9) imply that

$$\lambda v'\left(x\left(\theta^{n},z\right)\right)\geq u'\left(\omega-i\left(z\right)+f\left(i\left(z\right)\right)+\theta^{n}-x\left(\theta^{n},z\right)\right), \tag{Appendix D.16}$$

which is a strict inequality only if $x\left(\theta^{n},z\right)=f\left(i\left(z\right)\right)+\theta^{n}$. It is easy to verify that constraint (*AppendixD*.11) is satisfied under such a solution. This is because the value of $x\left(\theta^{n},z\right)$ for which (*AppendixD*.16) binds is such that $x\left(\theta^{n},z\right)-\theta^{n}$ is strictly declining in θ^{n}, where this follows from the concavity of $v\left(\cdot\right)$ and $u\left(\cdot\right)$. Consequently, if (*AppendixD*.16) is a strict inequality with $x(\theta^{n},z)=f\left(i\left(z\right)\right)+\theta^{n}$ for some n, then it follows that $x\left(\theta^{n-k},z\right)=f\left(i\left(z\right)\right)+\theta^{n-k}$ for all $k<n-1$. It therefore follows that there exists some n^{*} (which could be 1 or N) such that $x\left(\theta^{n},z\right)=f\left(i\left(z\right)\right)+\theta^{n}$ if $n<n^{*}$ and $x\left(\theta^{n},z\right)<f\left(i\left(z\right)\right)+\theta^{n}$ if $n\geq n^{*}$ with $x\left(\theta^{n},z\right)-\theta^{n}$ is

strictly declining in θ^n if $n \geq n^*$. Therefore, this means that (*AppendixD*.11) is satisfied under this solution.

Note that given the strict concavity of the program and the convexity of the constraint set with respect to $c(\theta^n, z)$, $x(\theta^n, z)$, and $i(z)$ since (*AppendixD*.12) is ignored, it follows that these values are uniquely defined conditional on λ. Moreover, since $J'(V^+(\theta^n, z)) = -\lambda \ \forall n$, it follows by forward iteration on the recursive program that the same $c(\theta^n, z)$, $x(\theta^n, z)$, and $i(z)$ are chosen at all future dates. Therefore, $V^+(\theta^n, z) = V$. Given that $C_{n+1,n} = 0$ under this solution, this can only be true if $x\left(\theta^{n+1}, z\right) = x(\theta^n, z) + \theta^{n+1} - \theta^n \ \forall n < N$, which given the above reasoning is only true if $x(\theta^n, z) = f(i(z)) + \theta^n \ \forall n$. If that is the case, then the welfare of households given V equals $u(\omega - i(z)) / (1 - \beta)$, which means that for (*AppendixD*.13) to be satisfied, it must be the case that $i(z) = 0$, but this violates part (ii) of Lemma 9 in Appendix Appendix B.

Case 2. Now suppose that α varies with the value of z. We can show first that conditional on z, it must be that

$$J'\left(V^+\left(\theta^1, z\right)\right) > J'\left(\sum_{n=1}^{N} \pi(\theta^n)\left(v(x(\theta^n, z)) + \beta V^+(\theta^n, z)\right)\right). \qquad \text{(Appendix D.17)}$$

To see why, note that if such an allocation is instead chosen for all z, it provides a continuation value

$$V = \sum_{n=1}^{N} \pi(\theta^n)\left(v(x(\theta^n, z)) + \beta V^+(\theta^n, z)\right) \qquad \text{(Appendix D.18)}$$

to the policymaker and optimality requires that it provides social welfare equal to

$$J\left(\sum_{n=1}^{N} \pi(\theta^n)\left(v(x(\theta^n, z)) + \beta V^+(\theta^n, z)\right)\right). \qquad \text{(Appendix D.19)}$$

This is because exceeding this social welfare is not possible by definition and if the implied social welfare were below the above value, the original solution would be suboptimal. It follows then that given V which satisfies (*AppendixD*.18), case 1 can be applied, which means that (*AppendixD*.17) must be satisfied.

Now note that if α varies with the value of z, it must be that given $z' \neq z''$,

$$\begin{aligned} J'\left(\sum_{n=1}^{N} \pi(\theta^n)\left(v\left(x\left(\theta^n, z'\right)\right) + \beta V^+\left(\theta^n, z'\right)\right)\right) &= J'\left(\sum_{n=1}^{N} \pi(\theta^n)\left(v\left(x\left(\theta^n, z''\right)\right) + \beta V^+\left(\theta^n, z''\right)\right)\right) \\ &= J'(V). \end{aligned} \qquad \text{(Appendix D.20)}$$

This is because (12) and the weak concavity of $J(V)$ imply that

$$J(V) \geq \int_0^1 J\left(\sum_{n=1}^{N} \pi(\theta^n)\left(v(x(\theta^n, z)) + \beta V^+(\theta^n, z)\right)\right) dz, \qquad \text{(Appendix D.21)}$$

and analogous arguments as used previous imply that conditional on z, social welfare must equal ($AppendixD.19$) which means that ($AppendixD.21$) must hold as an equality. However, if the value of

$$\sum_{n=1}^{N} \pi\left(\theta^n\right)\left(v\left(x\left(\theta^n, z\right)\right) + \beta V^+\left(\theta^n, z\right)\right)$$

varies with z, for ($AppendixD.21$) to hold as an equality, it must be that ($AppendixD.20$) also holds. Combining ($AppendixD.17$) with ($AppendixD.20$), it follows that $J'\left(V^+\left(\theta^1, z\right)\right) > J'(V)\ \forall z$. ■

Lemma 14. *If $V = \overline{V}$, then the solution to program P_0 admits $V^+\left(\theta^1, z\right) < V\ \forall z$.*

Proof. Suppose that $V = \overline{V}$. Lemma 6 in Appendix Appendix B implies that the solution admits $P(z) = 1\ \forall z$; part (i) of Lemma 9 in Appendix Appendix B implies that the household participation constraint (17) binds $\forall z$; and the arguments in the proof of part (i) of Lemma 9 in Appendix Appendix B imply that the policymaker achieves a continuation value of $\overline{V}\ \forall z$.

Suppose it were the case that for some z, $V^+\left(\theta^1, z\right) \geq V$, which given part (i) of Lemma 7 in Appendix Appendix B as well as the feasibility constraint (18) implies that $V^+\left(\theta^n, z\right) = \overline{V}\ \forall n$. This implies that $J\left(V^+\left(\theta^n, z\right)\right) = u(\omega)/(1-\beta)$ by part (i) of Lemma 9 in Appendix Appendix B. This also implies from the truth-telling constraint (15) that $x\left(\theta^n, z\right) = x\left(\theta^{n+1}, z\right) + \theta^{n+1} - \theta^n$ $\forall n < N$. Therefore, since (17) binds we have that $c\left(\theta^n, z\right) = \omega\ \forall n$. Together with the fact that $\omega - i(z) + f(i(z)) \leq \omega - i^* + f(i^*)$ this means that $x\left(\theta^n, z\right) \leq f(i^*) - i^* + \theta^n$. We now show that this previous relationship must hold with equality $\forall n$. Suppose it were the case that $x\left(\theta^1, z\right) < f(i^*) - i^* + \theta^1$. Then this would imply from part (i) of Lemma 7 in Appendix Appendix B that $x\left(\theta^n, z\right) < f(i^*) - i^* + \theta^n$ so that

$$\overline{V} = \sum_{n=1}^{N} \pi\left(\theta^n\right) v\left(x\left(\theta^n, z\right)\right)/(1-\beta) < \sum_{n=1}^{N} \pi\left(\theta^n\right) v\left(f(i^*) - i^* + \theta^n\right)/(1-\beta). \tag{Appendix D.22}$$

However, given the arguments in the proof of Lemma 2 in the text, there exists an equilibrium which provides a welfare equal to the right hand side of ($AppendixD.22$). This however contradicts the fact that $\overline{V}$ corresponds to the highest equilibrium welfare for the policymaker. Therefore, it is necessary that $x\left(\theta^n, z\right) = f(i^*) - i^* + \theta^n\ \forall n$. Note that since conditional on z, the policymaker receives $\overline{V}$ whereas households receive $u(\omega)/(1-\beta)$, a solution for which the elements of α do not depend on the realization of z exists.

We can focus on such a solution and we can show that this solution is suboptimal because it is possible to make households strictly better off while leaving the policymaker as well off. In order to establish this, we first establish the following upper bound on $J'\left(\overline{V}\right)$ which must hold given that $x\left(\theta^n, z\right) = f(i^*) - i^* + \theta^n$ and $V^+\left(\theta^n, z\right) = \overline{V}$ for all n and all z at $V = \overline{V}$.

Define $J'\left(\overline{V}\right) = \lim_{\epsilon\to 0^+}\left(J\left(\overline{V}\right) - J\left(\overline{V}-\epsilon\right)\right)/\epsilon$. Then it must be that

$$J'\left(\overline{V}\right) \leq -u'(\omega) / \left(\sum_{n=1}^{N} \pi(\theta^n) v'(f(i^*) - i^* + \theta^n)\right). \qquad \text{(Appendix D.23)}$$

To see why this is the case, consider the following potential solution starting from $V = \overline{V} - \epsilon$ for $\epsilon > 0$ arbitrarily small. Let $P(z) = 1\ \forall z$, $V^+(\theta^n, z) = \overline{V}\ \forall n, z$, and $i(z) = i^*\ \forall z$. Moreover, $\forall n, z$, let $x(\theta^n, z) = f(i^*) - i^* + \theta^n - \varepsilon(\epsilon)$ for $\varepsilon(\epsilon)$ which satisfies

$$\epsilon = \sum_{n=1}^{N} \pi(\theta^n)\left(v(f(i^*) - i^* + \theta^n) - v(f(i^*) - i^* + \theta^n - \varepsilon(\epsilon))\right). \qquad \text{(Appendix D.24)}$$

It is straightforward to verify that the conjectured solution satisfies all of the constraints of the problem for sufficiently small ϵ. Moreover, since such a solution is always feasible, it implies that

$$\frac{J\left(\overline{V}-\epsilon\right) - J\left(\overline{V}\right)}{\epsilon} \geq \frac{u(\omega + \varepsilon(\epsilon)) - u(\omega)}{\epsilon}.$$

Taking the limit of both sides of the above inequality as ϵ approaches 0 implies ($Appendix D.23$).

Given the bound in (Appendix D.23), we can now show that the proposed solution at $V = \overline{V}$ is suboptimal. To see why, consider the following perturbation. Suppose that $x(\theta^n, z)$ were increased by $dx^n = \epsilon > 0$ arbitrarily small for all $n < N$. Moreover, suppose that $V^+(\theta^n, z)$ were reduced by some $dV^n(\epsilon)$ which satisfies

$$dV^n(\epsilon) = \frac{1}{\beta}\sum_{n=1}^{N} \pi(\theta^n)\left(v(f(i^*) - i^* + \theta^n + \epsilon) - v(f(i^*) - i^* + \theta^n)\right) \qquad \text{(Appendix D.25)}$$

for all $n < N$. Finally, suppose that $x\left(\theta^N, z\right)$ were decreased by some $dx^N(\epsilon)$ which satisfies:

$$\begin{aligned} v(f(i^*) - i^* + \theta^N - dx^N(\epsilon)) &= v(f(i^*) - i^* + \theta^N + \epsilon) \\ &\quad - \sum_{n=1}^{N} \pi(\theta^n)\left(v(f(i^*) - i^* + \theta^n + \epsilon) - v(f(i^*) - i^* + \theta^n)\right). \end{aligned}$$

Note that from the above $dx^N(\epsilon) > 0$. It can be verified that the proposed perturbation continues to satisfy all of the constraints of the problem. In order that this perturbation not strictly increase the welfare of households as ϵ approaches 0, it must be that:

$$-\sum_{n=1}^{N-1} \pi(\theta^n) u'(\omega) + \pi\left(\theta^N\right) u'(\omega) \lim_{\epsilon>0, \epsilon\to 0} \frac{dx^N(\epsilon)}{\epsilon} - \beta \sum_{n=1}^{N-1} \pi(\theta^n) J'\left(\overline{V}\right) \lim_{\epsilon>0,\epsilon\to 0} \frac{dV^n(\epsilon)}{\epsilon} \leq 0$$

Substituting ($Appendix D.23$) and ($Appendix D.25$) into the above implies that
$\pi(\theta^N)u'(\omega)\lim_{\epsilon>0,\epsilon\to 0}\frac{dx^N(\epsilon)}{\epsilon} \leq 0$, which is a contradiction since $dx^N(\epsilon)$ is strictly positive. ■

We now go back to the proof of part (ii) and use the two previous lemma to show that $\underline{V}'$ does not exist so as to create a contradiction.

Step 1. Note that if $V \in \Psi$ so that the solution to program P_0 given V admits $i(z) = i^*$, it must be the case that $P(z) = 1\ \forall z$, that $V > V_0$, and that the limited commitment constraint (16) is slack $\forall z$ so that Lemmas 13 and 14 can apply. To see why, suppose first it were the case that $P(z) = 0$ for some positive measure z. Then Lemma 6 in Appendix Appendix B would imply that $V < V_0$, leading to a contradiction of the fact that $\underline{V}^+ \geq V_0$. Furthermore, suppose it were the case that (16) is not slack so that the Lagrange multiplier $\psi(\theta^n, z) > 0$ for some some n. Then one can apply the arguments of step 2 in the proof of part (i) to show that this would imply that $i(z) < i^*$. Therefore, $\psi(\theta^n, z) = 0\ \forall n$ so that (16) is slack.

Step 2. Starting from any given $V_{t+k} \in \Psi$, $\Pr\{V_{t+k+1} < V_{t+k}\} \geq \pi(\theta^1)$. If $V_{t+k} = \overline{V}$, then this follows directly from Lemma 14 since $V_{t+k+1} < V_{t+k}$ if $\theta_{t+k} = \theta^1$. If instead $V_{t+k} \in (V_0, \overline{V})$, it then follows from the arguments in Lemma 13. Therefore, in this case, $V_{t+k+1} < V_{t+k}$ if $\theta_{t+k} = \theta^1$.

Step 3. We show in this step that for any $V_{t+k} \in \Psi$, there exists a finite l such that if state 1 is repeated l times consecutively, then $V_{t+k+l} < \underline{V}^+$. This step thus contradicts the fact that $\underline{V}^+$ is the infinum of long run continuation values in Ψ, which completes the proof. To see why this is true, let $g(V)$ correspond to the highest realization of $V^+(\theta^1, z) \in \Psi$ in the solution to the problem with state V. $g(V)$ is a continuous correspondence given the continuity of the objective function and the compactness and continuity of the constraint set. It follows that in a sequence under which θ^1 is repeated l times, $V_{t+k+l} \leq g(V_{t+k+l-1}) < V_{t+k+l-1}$, where we have used step 2. Suppose by contradiction that there is no finite l such that $V_{t+k+l} < \underline{V}^+$. This means that $\lim_{l\to\infty} V_{t+k+l} = V^+ \geq \underline{V}^+$ for some $V^+ \in \Psi$. This implies that $\lim_{l\to\infty} g(V_{t+k+l-1}) \geq V^+ \geq \underline{V}^+$. However, given the continuity of $g(\cdot)$, this implies that $g(V^+) \geq V^+$, which contradicts the fact that $g(V) < V$ for all $V \in \Psi$.■

Appendix D.2. Proof of Proposition 3

Suppose that (24) holds and suppose that $\Pr\{\lim_{t\to\infty} P_t = 1\} > 0$, so that a permanent dictator emerges following some positive measure of histories. Let $\{V_t\}_{t=0}^{\infty}$ correspond to the stochastic sequence of continuation values under some such history, which must includes within it at least one convergent stochastic subsequence, where this follows from the fact that V_t is bounded. With some abuse of notation, let Ψ correspond to the set of all limiting values of convergent stochastic subsequences of $\{V_t\}_{t=0}^{\infty}$. It follows that $V \in \Psi$, that the solution to program P_0 admits $V^+(\theta, z) \in \Psi\ \forall\theta$ and $\forall z$ with positive measure since $V^+(\theta, z)$ must itself be within the set of limits of stochastic subsequences. In order to proceed, we establish that for any $V \in \Psi$, $i(z) = i^*\ \forall z$. After this is established, the arguments in part (ii) of the proof of Proposition 2 can be used to show that this is not possible, completing the proof.

Given V, the truth-telling constraint (15) for $\widehat{\theta} = \theta^1$ together with the fact that $x(\theta^1, z) \geq 0$

implies that conditional on z and θ,

$$v\left(x\left(\theta^{n},z\right)\right)+\beta V^{+}\left(\theta^{n},z\right)\geq v(\theta^{n}-\theta^{1})+\beta V^{+}\left(\theta^{n},z\right). \quad \text{(Appendix D.26)}$$

Since replacement never takes place in the future, forward iteration on (*AppendixD*.26) implies that

$$v\left(x\left(\theta^{n},z\right)\right)+\beta V^{+}\left(\theta^{n},z\right)\geq v(\theta^{n}-\theta^{1})+\beta\frac{\sum_{l=1}^{N}\pi\left(\theta^{l}\right)v\left(\theta^{l}-\theta^{1}\right)}{1-\beta}. \quad \text{(Appendix D.27)}$$

Together with (24), (*AppendixD*.27) implies that the limited commitment constraint (16) holds as a strict inequality for all θ^n. To see why, from part (iv) of Lemma 7 in Appendix Appendix B, it is sufficient to verify that (16) is slack if $\theta=\theta^1$. Given that $i(z)\leq i^*$ from Lemma 12, it is clear that this is the case given (24) and (*AppendixD*.27). Since (16) holds as a strict inequality for all θ^n, the Lagrange multiplier $\psi(\theta^n,z)$ for constraint (16) must equal zero, which implies from (*AppendixD*.3) that $i(z)=i^*$ $\forall z$. This establishes that if a permanent dictator emerges following some positive measure of histories, then the stochastic sequence $\{i_t\}_{t=0}^{\infty}$ converges to i^* for a positive measure of histories. However, this contradicts part (ii) of Proposition 2.■

Appendix D.3. Proof of Proposition 4

In order to prove this result, we establish the following preliminary lemmas.

Lemma 15. *$\exists V^*\in[V_0,\overline{V})$ s.t. if $V\geq V^*$, the solution to program P_0 admits $i(z)=i^*$ $\forall z$.*

Proof. We first establish that if $V=\overline{V}$, then the limited commitment constraint (16) is a strict inequality and $i(z)=i^*$. Consider the solution given $V=\overline{V}$. We can use the same arguments as in the proof of Lemma 8 in Appendix Appendix B so as to perturb the solution so as to let $C_{n+1,n}=0$, preserving the level of investment. The same arguments as those used in the proof of Lemma 14 imply that there is no replacement and policymakers receives a continuation value of $\overline{V}$ conditional on the realization of z. Using the truth-telling constraint (15), it follows that for any $n^*\in\{1,...,N\}$,

$$\begin{aligned}\overline{V} &= \sum_{n=1}^{N}\pi\left(\theta^{n}\right)\left(v\left(x\left(\theta^{n},z\right)\right)+\beta V^{+}\left(\theta^{n},z\right)\right)\\ &\leq \sum_{n=1}^{n^*}\pi\left(\theta^{n}\right)\left(v(x(\theta^{n^*},z))+\beta V^{+}(\theta^{n^*},z)\right)+\sum_{n=n^*+1}^{N}\pi\left(\theta^{n}\right)\left(v\left(x\left(\theta^{n},z\right)\right)+\beta V^{+}\left(\theta^{n},z\right)\right).\end{aligned}$$

Using part (i) of Lemma 7 in Appendix Appendix B together with the fact that $V^{+}\left(\theta^{n^*},z\right)\leq\overline{V}$, it follows that the above condition reduces to

$$\overline{V}\leq\frac{\sum_{n=1}^{n^*}\pi\left(\theta^{n}\right)v\left(x\left(\theta^{n^*},z\right)\right)+\sum_{n=n^*+1}^{N}\pi\left(\theta^{n}\right)v\left(x\left(\theta^{n^*},z\right)+\theta^{n}-\theta^{n^*}\right)}{1-\beta}. \quad \text{(Appendix D.28)}$$

Given Lemma 2, it is necessary that

$$\overline{V} \geq \frac{\sum_{n=1}^{N} \pi\left(\theta^{n}\right) v\left(f\left(i^{*}\right)-i^{*}+\theta^{n}\right)}{1-\beta}, \tag{Appendix D.29}$$

since an equilibrium providing the right hand side of (*AppendixD*.29) to the incumbent exists. (*AppendixD*.28) and (*AppendixD*.29) thus imply that

$$x(\theta^{n}, z) \geq f\left(i^{*}\right)-i^{*}+\theta^{1} \forall n. \tag{Appendix D.30}$$

Now consider the value of $\overline{V}$. Suppose that the limited commitment constraint (16) holds as an equality, where by part (iv) of Lemma 7 in Appendix Appendix B, this must be the case for $\theta^{n}=\theta^{1}$. Taking into account that (16) holds as an equality together with the fact that local downward informational constraints bind, it is the case that $\overline{V}$ can be rewritten as

$$\overline{V}=v(f\left(i\left(z\right)\right)+\theta^{1})+\beta \underline{V}+\sum_{n=2}^{N}\left(\sum_{l=2}^{N} \pi(\theta^{l})\right)\left(v(x(\theta^{n-1}, z)+\theta^{n}-\theta^{n-1})-v(x(\theta^{n-1}, z))\right). \tag{Appendix D.31}$$

(*AppendixD*.29) and (*AppendixD*.30) together with the fact that $v\left(\cdot\right)$ is strictly concave and $i\left(z\right) \leq i^{*}$ from Lemma 12 imply that (*AppendixD*.31) simplifies to

$$\frac{\sum_{n=1}^{N} \pi\left(\theta^{n}\right) v\left(f\left(i^{*}\right)-i^{*}+\theta^{n}\right)}{1-\beta} \leq v(f\left(i^{*}\right)+\theta^{1})+\beta \underline{V}+\Gamma$$

for Γ defined in (26). However, if $N=2$, it can be shown by some algebra that this violates Assumption 2 and if $N>2$, this violates (25).

Therefore, if $V=\overline{V}$, then the solution admits (16) as a strict inequality and therefore $i\left(z\right)=i^{*}$. By standard arguments, the solution is continuous in V, which implies that if (16) is a strict inequality in the solution to program P_0 given $V=\overline{V}$, then there exists some $V^{*}<\overline{V}$ such that (16) is a strict inequality in the solution to program P_0 given $V>V^{*}$ and therefore $i\left(z\right)=i^{*}$. ■

Lemma 16. *Consider the solution to program P_0 given $V \in [V_0, \overline{V})$. Then it must be that the solution admits either $V^{+}\left(\theta^{N}, z\right)=\overline{V}$ or $J'\left(V^{+}\left(\theta^{N}, z\right)\right)<J'\left(V\right) \forall z$.*

Proof. We pursue a similar strategy to that used in the proof of Lemma 13. Consider first the case in which the solution α does not vary with the realized value of z. Suppose that $V^{+}\left(\theta^{N}, z\right)<\overline{V}$ and suppose by contradiction that $J'\left(V^{+}\left(\theta^{N}, z\right)\right) \geq J'(V) \forall z$. Part (i) of Lemma 7 in Appendix Appendix B together with the weak concavity of $J\left(V\right)$ then implies that $J'\left(V^{+}\left(\theta^{n}, z\right)\right) \geq J'(V) \forall n$ and $\forall z$, where we have used the fact that α does not depend on z. From Lemma 8 in Appendix Appendix B, one can perturb such a solution without changing households' welfare and continuing to satisfy the constraints of the problem by changing the values of $V^{+}\left(\theta^{n}, z\right)$ so that $C_{n+1,n}=0$ for all $n<N$. Note that this perturbation weakly decreases the value of $V^{+}\left(\theta^{N}, z\right)$ so that it remains the case that $J'\left(V^{+}\left(\theta^{N}, z\right)\right) \geq J'(V) \forall z$.

Now consider such a solution. ($Appendix D.4$) together with ($Appendix D.5$) then imply that

$$\sum_{n=1}^{N} \pi\left(\theta^{n}\right) J'\left(V^{+}\left(\theta^{n}, z\right)\right) = J'\left(V\right) - \pi(\theta^{1})\psi(\theta^{1}, z) - \pi(\theta^{1})\underline{\mu}(\theta^{1}, z) \qquad \text{(Appendix D.32)}$$

where we have used the fact that $\eta\left(z\right) = 0$ since otherwise the household participation constraint (17) binds $\forall z$ so that $J\left(V\right) = u\left(\omega\right) / \left(1 - \beta\right)$, and where we have used Lemma 7 in Appendix Appendix B together with the fact that $J'\left(V^{+}\left(\theta^{n}, z\right)\right) \geq J'(V)\ \forall n$. It thus follows from the above condition that $\psi\left(\theta^{1}, z\right) = 0$ so that (16) can be ignored. Analogous arguments as in the proof of Lemma 13 can then apply and be used to argue that in this case that $J'\left(V^{+}\left(\theta^{N}, z\right)\right) < J'\left(V\right)\ \forall z$, leading to a contradiction. Moreover, analogous arguments as those used in the proof of Lemma 13 for the case in which α varies with z also apply. ■

We now go back to the proof of Proposition 4.

Proof of part (i). We begin by looking at investment dynamics and proceed in an analogous fashion as in the proof of Proposition 2. Suppose it were the case that $\Pr\left\{\lim_{t\to\infty} i_t = \widehat{i}\right\} > 0$ for some $\widehat{i}$. By Proposition 2, it is necessarily the case that $\widehat{i} \neq i^{*}$. This would imply that there exist a stochastic sequence $\{i_t\}_{t=0}^{\infty}$ with positive measure which converges to $\widehat{i}$. Associated with such a stochastic sequence is a stochastic sequence $\{V_t\}_{t=0}^{\infty}$ which must includes within it at least one convergent stochastic subsequence, where this follows from the fact that V_t is bounded. Let Ψ correspond to the entire set of all limiting values of convergent stochastic subsequences of $\{V_t\}_{t=0}^{\infty}$ for every single stochastic sequence $\{i_t\}_{t=0}^{\infty}$ which converges to $\widehat{i}$. It follows by the continuity of the policy function that the solution to program P_0 starting from some $V \in \Psi$ yields a solution which admits $i\left(z\right) = \widehat{i}\ \forall z$. Moreover, it must be that $V^{+}\left(\theta, z\right) \in \Psi\ \forall \theta$ and $\forall z$ with positive measure since $V^{+}\left(\theta, z\right)$ must itself be within the set of limits of stochastic subsequences. Let $\overline{V}^{+}$ correspond to the supremum of continuation values in Ψ. It is clear that $\overline{V}^{+} < V^{*}$ for V^{*} defined in Lemma 15, since if this were not the case, this would imply that $i\left(z\right) = i^{*} \neq \widehat{i}$ for some $V \in \Psi$, leading to a contradiction. The strategy of our proof is to show that $\overline{V}'$ does not exist so as to create a contradiction. We proceed in the following steps.

Step 1. Starting from any given $V_{t+k} \in \Psi$, $\Pr\left\{V_{t+k+1} > V_{t+k}\right\} \geq \pi\left(\theta^{N}\right)$. If $V_{t+k} \in [V_0, \overline{V})$, then this follow directly from Lemma 16. If instead $V_{t+k} < V_0$, then independently of the realization of z, the policymaker choosing policies–whether it is the previous incumbent or the new replacement policymaker–receives an expected continuation value equal to V_0. As such, the arguments of Lemma 16 starting from $V = V_0$ establish that $V_{t+k+1} > V_0 > V_{t+k}$ if $\theta_{t+k} = \theta^{N}$.

Step 2. We show in this step that for any $V_{t+k} \in \Psi$, there exists a finite l such that if state 1 is repeated l times consecutively, then $V_{t+k+l} > \overline{V}^{+}$. This step thus contradicts the fact that $\overline{V}^{+}$ is the supremum of long run continuation values in Ψ, which completes the proof. To see why this is true, with some abuse of notation, let $g\left(V\right)$ correspond to the lowest realization of $V^{+}\left(\theta^{N}, z\right) \in \Psi$ in the solution to the problem with state V. $g(V)$ is a continuous correspondence given the continuity of the objective function and the compactness and continuity of the constraint set. It follows that in a sequence under which θ^N is repeated l times, $V_{t+k+l} \geq g\left(V_{t+k+l-1}\right) > V_{t+k+l-1}$, where we have used step 1. Suppose by contradiction that there is no finite l such that $V_{t+k+l} > \overline{V}^+$. This means that $\lim_{l\to\infty} V_{t+k+l} = V^+ \leq \overline{V}^+$ for some $V^+ \in \Psi$. This implies that $\lim_{l\to\infty} g\left(V_{t+k+l-1}\right) \leq V^+ \leq \overline{V}^+$. However, given the continuity of $g(\cdot)$, this implies that $g\left(V^+\right) \leq V^+$, which contradicts the fact that $g(V) > V$ for all $V \in \Psi$. This completes the proof that $\Pr\left\{\lim_{t\to\infty} i_t = \widehat{i}\right\} = 0$.

Proof of part (ii). Proving the second part of the proposition is straightforward given the dynamics of continuation values. Suppose it were the case that $\Pr\left\{\lim_{t\to\infty} x_t(\theta) = \widehat{x}(\theta)\right\} = 1$ for some $\widehat{x}(\theta)$. Using analogous reasoning as in the proof of part (i), one can define, with some abuse of notation, the set Ψ of limiting continuation values. It necessarily follows that the solution to program P_0 starting from some $V \in \Psi$ yields a solution which admits $x(\theta, z) = \widehat{x}(\theta)$ $\forall\theta$ and $\forall z$. Moreover, it must be that $V^+(\theta, z) \in \Psi$ $\forall\theta$ and $\forall z$ with positive measure since $V^+(\theta, z)$ must itself be within the set of limits of stochastic subsequences. This leaves two cases to consider.

Case 1. Suppose that $\Pr\left\{\lim_{t\to\infty} P_t = 1\right\} = 1$ so that there is no replacement in the long run. Given the promise-keeping constraint (12), this requires $\Pr\left\{\lim_{t\to\infty} V_t = \widehat{V}\right\} = 1$ for some $\widehat{V} > V_0$ so that Ψ is a singleton. However, Lemmas 13 and 14 together imply that $\Pr\left\{V_{t+k+1} < V_{t+k}\right\} \geq \pi\left(\theta^1\right)$ so that $\{V_t\}_{t=0}^{\infty}$ cannot converge to a single point since it must necessarily decrease.

Case 2. Suppose that $\Pr\left\{\lim_{t\to\infty} P_t = 1\right\} < 1$ so that there is replacement in the long run. We can show that in this case, it must be that $\sum_{n=1}^{N} \pi(\theta^n)\left(v\left(x\left(\theta^n, z\right)\right) + \beta V^+\left(\theta^n, z\right)\right) = V_0$ starting from all $V_{t+k} \in \Psi$. To see why, suppose this were not the case and consider the solution to program P_0 starting from some $V \in \Psi$. Use Lemma 8 in Appendix Appendix B so as to perturb such a solution without changing households' welfare and continuing to satisfy the constraints of the problem by changing the values of $V^+(\theta^n, z)$ so that $C_{n+1,n} = 0$ for all $n < N$. Given that $x(\theta, z) = \widehat{x}(\theta)$ $\forall\theta$ and $\forall z$, it follows that satisfaction of the promise-keeping constraint (12) and the truth-telling constraint (15) uniquely determines $\{V^+(\theta^n, z)\}_{n=1}^{N}$ $\forall z$, where $V^+(\theta^n, z)$ is strictly increasing in V. From (16) and (*AppendixD*.3), it follows that $i(z)$ is weakly larger the larger the value of $V^+\left(\theta^1, z\right)$ which means that $i(z)$ is weakly increasing in V. Lemma 12 implies that $i(z) \leq i^*$, which combined with the resource constraint (13), furthermore implies that the value of $\sum_{n=1}^{N} \pi(\theta^n)\, u\left(c\left(\theta^n, z\right)\right)$ is weakly increasing in V. However, combined with the fact that $V^+(\theta^n, z)$ is strictly increasing in V, this means that for $J(V)$ to be non-increasing in the range of $V \in \Psi$–which is established in Lemma 5 in Appendix Appendix B–it must be that the set Ψ is either a singleton, which is ruled out in case 1, or that $\Psi \subset [\underline{V}, V_0]$, in which case $\sum_{n=1}^{N} \pi(\theta^n)\left(v\left(x\left(\theta^n, z\right)\right) + \beta V^+\left(\theta^n, z\right)\right) = V_0$. However, the arguments in step 1 in the proof of part (i) imply that starting from $V_{t+k} = V_0$, $\Pr\left\{V_{t+k+1} > V_{t+k}\right\} \geq \pi\left(\theta^N\right)$, which means that it is not possible for $\Psi \subset [\underline{V}, V_0]$.■